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# COMPLETE FINSLER–RIEMANN SYSTEMS

*By* J. G. FREEMAN (*Bradford*)

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## 1. Introduction

A Finsler–Riemann system, consisting of an $n$-dimensional deformable subspace in a Riemann $(2n-1)$-dimensional space, together with trajectories and generators, has first fundamental form given by (**2**) (2.9), viz.

$$ds^2 = g_{ab}(x,u)\,dx^a dx^b \quad (a, b = 1, 2,..., n), \tag{1.1}$$

and the length $L$ of the element of support is given by

$$L^2 = g_{ab}\,u^a u^b. \tag{1.2}$$

The Finsler space having the $x^a$ for coordinates, the $u^a$ for components of the element of support, and the same function $L$ for the length of the element of support, will be called the *Finsler image* of the F–R system.

Those elements of the Finsler image which may differ from the corresponding elements of the F–R system will be distinguished by a vertical bar placed after them (but before the affixes), so that the first fundamental form of the image will be denoted by

$$ds^2| = g|_{ab}\,dx^a dx^b \tag{1.3}$$

and the absolute differential in the image of a vector with components $X^a$ by

$$DX|^a = dX^a + X^b\,dx^c\,\Gamma|_b{}^a{}_c + X^b\,du^c C|_b{}^a{}_c. \tag{1.4}$$

With this notation a *normal* F–R system as defined in (**2**) § 2 is one for which $g_{ab} = g|_{ab}$. If $C_b{}^a{}_c = C|_b{}^a{}_c$, the F–R system will be called *C-type*, if $\Gamma_b{}^a{}_c = \Gamma|_b{}^a{}_c$, it will be called $\Gamma$-*type*, and, if $g_{ab}$, $C_b{}^a{}_c$, $\Gamma_b{}^a{}_c$ equal $g|_{ab}$, $C|_b{}^a{}_c$, $\Gamma|_b{}^a{}_c$ respectively, it will be called *complete*, when also

$$\Gamma^*{}_b{}^a{}_c = \Gamma^*|_b{}^a{}_c, \qquad A_b{}^a{}_c = A|_b{}^a{}_c,$$

since the formulae [(**2**) (3.18), (3.19), (3.20)] expressing $\Gamma^*{}_b{}^a{}_c$, $A_b{}^a{}_c$ in terms of $\Gamma_b{}^a{}_c$, $C_b{}^a{}_c$ are then identical with the corresponding formulae for the image.

In a complete F–R system the $g_{ab}$, $\Gamma_b{}^a{}_c$, $C_b{}^a{}_c$, $\Gamma^*{}_b{}^a{}_c$, $A_b{}^a{}_c$ can be expressed in terms of derivatives of $L^2$ only (this being a property of the corresponding expressions in the image).

In the last section I consider whether there exists a complete F–R system having a given function for the square of the length of its element of support.

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## 2. Transformation of $S_n$-coordinates

When the coordinates of $S_n$ are transformed by equations of form

$$x^a = x^a(x''^1, x''^2, ..., x''^n), \tag{2.1}$$

the elements of the F–R system will be denoted by double dashes, so that the first fundamental form becomes

$$ds^2 = g''_{ab}(x'', u'')\,dx''^a dx''^b \tag{2.2}$$

and the absolute differential of a vector with transformed components $X''^a$ becomes

$$DX''^a = dX''^a + X''^b\,dx''^c\,\Gamma''{}_b{}^a{}_c + X''^b\,du''^c C''{}_b{}^a{}_c. \tag{2.3}$$

Put

$$\left.\begin{aligned} \partial x''^a/\partial x^b &= \nabla^a_b, & \partial x^a/\partial x''^b &= \Delta^a_b \\ \partial^2 x''^a/\partial x^b \partial x^c &= \nabla^a_{bc}, & \partial^2 x^a/\partial x''^b\,\partial x''^c &= \Delta^a_{bc} \end{aligned}\right\}, \tag{2.4}$$

so that $\nabla^a_b\,\Delta^c_a = \delta^c_b, \qquad \nabla^b_a\,\Delta^a_c = \delta^b_c.$

Since, by definition, the $DX^b$ are components of a vector,

$$DX''^a = \nabla^a_b\,DX^b.$$

By means of (2.3) and (2.4) this can be written

$$X''^e\,dx''^f[\Gamma''{}_e{}^a{}_f - \Delta^b_{ef}\,\nabla^a_b - \nabla^a_b\,\Delta^c_e\,\Delta^g_f\,\Gamma_c{}^b{}_g - \nabla^a_b\,\Delta^c_e\,\Delta^g_{hf}\,u''^h C_c{}^b{}_g] +$$
$$+ X''^e\,du''^f[C''{}_e{}^a{}_f - \nabla^a_b\,\Delta^c_e\,\Delta^g_f\,C_c{}^b{}_g] = 0.$$

Since the $X''^e$, $dx''^f$, $du''^f$ are arbitrary, it follows that

$$\Gamma''{}_e{}^a{}_f = \Delta^c_e\,\nabla^a_b\,\Delta^g_f\,\Gamma_c{}^b{}_g + \Delta^b_{ef}\,\nabla^a_b + \Delta^c_e\,\nabla^a_b\,\Delta^g_{hf}\,\nabla^h_j\,u^j C_c{}^b{}_g \tag{2.5}$$

and

$$C''{}_e{}^a{}_f = \Delta^c_e\,\nabla^a_b\,\Delta^g_f\,C_c{}^b{}_g. \tag{2.6}$$

Since an identical method may be used to establish the corresponding transformation formulae for the Finsler image, it follows that $C_c{}^b{}_g$, $\Gamma_c{}^b{}_g$, $C''{}_e{}^a{}_f$, $\Gamma''{}_e{}^a{}_f$ in (2.5), (2.6) can be replaced by $C|_c{}^b{}_g$, $\Gamma|_c{}^b{}_g$, $C''|_e{}^a{}_f$, $\Gamma''|_e{}^a{}_f$ respectively.

Since also

$$g''_{ab} = \Delta^e_a\,\Delta^f_b\,g_{ef}, \tag{2.7}$$

with a similar formula in $g|_{ef}$ and $g''|_{ab}$, it follows that:

*If an F–R system is* (i) *normal, and/or* (ii) *C-type, and/or* (iii) *both C-type and Γ-type, it remains so after a transformation of the coordinates.*

Since equations (2.5) involve the $C_b{}^a{}_c$, if a system is Γ-type but not $C$-type, it will not, in general, remain Γ-type after transformation.

## 3. Simplification by change of $S_\nu$-coordinates

In certain cases it is convenient to transform the $S_\nu$-coordinates, taking as coordinate lines in $S_\nu$ those of $S_n$ and its trajectories, so that

$$y^a = x^a, \qquad y^A = t^A. \tag{3.1}$$

To avoid confusion in this case $g_{\alpha\beta}$, $\left\{\begin{matrix} \alpha \\ \beta\ \gamma \end{matrix}\right\}$, $[\alpha\beta, \gamma]$ will be replaced by $g'_{\alpha\beta}$, $\left\{\begin{matrix} \alpha \\ \beta\ \gamma \end{matrix}\right\}'$, $[\alpha\beta, \gamma]'$ respectively. Then

$$g_{ab} = g'_{\alpha\beta} B^\alpha_a B^\beta_b = g'_{\alpha\beta} \delta^\alpha_a \delta^\beta_b$$

from (3.1). Therefore

$$g_{ab} = g'_{ab}. \tag{3.2}$$

Also from (**2**) (3.9)

$$\begin{aligned} [bc, e] &= g_{ae}\left\{\begin{matrix} a \\ b\ c \end{matrix}\right\} \\ &= g_{ae} B^a_\alpha\left(B^\alpha_{bc} + B^\beta_b B^\gamma_c \left\{\begin{matrix} \alpha \\ \beta\ \gamma \end{matrix}\right\}'\right) \\ &= g_{ae} g^{af} g'_{\alpha\rho} B^\rho_f \delta^\beta_b \delta^\gamma_c \left\{\begin{matrix} \alpha \\ \beta\ \gamma \end{matrix}\right\}' \\ &= \delta^f_e \delta^\rho_f \delta^\beta_b \delta^\gamma_c [\beta\gamma, \rho]'. \end{aligned}$$

Therefore

$$[bc, e] = [bc, e]'. \tag{3.3}$$

Also

$$\left\{\begin{matrix} a \\ b\ c \end{matrix}\right\} = g^{ae}[bc, e].$$

Consequently

$$\left\{\begin{matrix} a \\ b\ c \end{matrix}\right\} = g^{ae}[bc, e]'. \tag{3.4}$$

Further, from (**2**) (3.10),

$$\begin{aligned} \left\{\begin{matrix} a \\ b\ A \end{matrix}\right\} &= B^a_\alpha\left(B^\alpha_{bA} + B^\beta_b B^\gamma_A \left\{\begin{matrix} \alpha \\ \beta\ \gamma \end{matrix}\right\}'\right) \\ &= g^{ae} g'_{\alpha\rho} B^\rho_e \delta^\beta_b \delta^\gamma_A \left\{\begin{matrix} \alpha \\ \beta\ \gamma \end{matrix}\right\}' \\ &= g^{ae} \delta^\rho_e \delta^\beta_b \delta^\gamma_A [\beta\gamma, \rho]'. \end{aligned}$$

Thus

$$\left\{\begin{matrix} a \\ b\ A \end{matrix}\right\} = g^{ae}[bA, e]'. \tag{3.5}$$

Since, from (**2**) (3.15),

$$\Gamma_b{}^a{}_c = \left\{\begin{matrix} a \\ b\ c \end{matrix}\right\} + \left\{\begin{matrix} a \\ b\ A \end{matrix}\right\} p^A_c$$

and

$$C_b{}^a{}_c = \left\{\begin{matrix} a \\ b\ A \end{matrix}\right\} q^A_c$$

from (3.2), (3.3), (3.4), (3.5), we now have

$$\left.\begin{aligned} \Gamma_{bac} &= [bc, a]' + [bA, a]' p^A_c \\ C_{bac} &= \qquad\qquad [bA, a]' q^A_c \end{aligned}\right\} \tag{3.6}$$

*when using $S_\nu$-coordinates for which $y^a = x^a$, $y^A = t^A$.*

## 4. Normal $C$-type systems

It will be proved that:

*Necessary and sufficient conditions for a system to be normal and $C$-type are*

$$\text{(i) } C_{0bc} = 0 \quad \textit{and} \quad \text{(ii) } C_{abc} = C_{bac}.$$

They are necessary: for, if the system is both normal and $C$-type, then

$$C_{abc} = g_{be} C_{a\ c}^{\ e} = g|_{be} C|_{a\ c}^{\ e} = C|_{abc}$$

and

$$C|_{0bc} = 0, \qquad C|_{abc} = C|_{bac}.$$

They are sufficient: for, using (ii) and (**2**) (4.4), we have

$$\partial g_{ab}/\partial u^c = C_{abc} + C_{bac} = 2C_{abc}. \tag{4.1}$$

From

$$L^2 = g_{ab} u^a u^b,$$

$$\begin{aligned}\partial L^2/\partial u^c &= (\partial g_{ab}/\partial u^c) u^a u^b + 2g_{ac} u^a \\ &= 2C_{abc} u^a u^b + 2g_{ac} u^a\end{aligned}$$

from (4.1). Hence, using (i), we get

$$\partial L^2/\partial u^c = 2g_{ac} u^a.$$

Also, from (4.1) and (i),

$$\begin{aligned}\partial^2 L^2/\partial u^c \partial u^b &= 2(\partial g_{ac}/\partial u^b) u^a + 2g_{bc} \\ &= 4C_{acb} u^a + 2g_{bc} \\ &= 2g_{bc}.\end{aligned}$$

Hence

$$g_{bc} = \tfrac{1}{2}\partial^2 L^2/\partial u^c \partial u^b = g|_{bc},$$

and the system is therefore normal; from (4.1),

$$C_{abc} = \tfrac{1}{2}\partial g_{ab}/\partial u^c = \tfrac{1}{2}\partial g|_{ab}/\partial u^c = C|_{abc},$$

and the system is therefore $C$-type.

It will also be proved that:

*If a system is normal, a necessary and sufficient condition for it to be $C$-type is $C_{abc} = C_{bac}$.*

It is proved necessary as above. It is sufficient; for then, from (**2**) (4.4),

$$C_{abc} = \tfrac{1}{2}(C_{abc} + C_{bac}) = \tfrac{1}{2}\partial g_{ab}/\partial u^c = \tfrac{1}{2}\partial g|_{ab}/\partial u^c = C|_{abc}.$$

## 5. System with element of support transported along trajectories by induced parallelism in $S_n$ with deformation of $S_\nu$

In this case $Dl^a = 0$ when $dx^a = 0$. From

$$Dl^a = dl^a + l^b dx^c \Gamma_{b\ c}^{\ a} + l^b du^c C_{b\ c}^{\ a},$$

$$u^a = Ll^a,$$

and (2) (3.17) we have

$$0 = dl^a + l^b(L\,dl^c + dL\,l^c)C_b{}^a{}_c$$
$$= dl^a + l^b L\,dl^c\,C_b{}^a{}_c,$$

i.e.
$$0 = dl^c \theta^a_c, \tag{5.1}$$

where
$$\theta^a_c = \delta^a_c + l^b L C_b{}^a{}_c. \tag{5.2}$$

The $dl^a$ cannot all vanish: for, if they did, the $l^a$ would be constant along trajectories, and the $t^A$ (expressible as functions of the $l^a$ and $x^a$) would, contrary to supposition, be constant along trajectories. Hence, from (5.1), the determinant
$$\theta = |\theta^a_c| = 0 \tag{5.3}$$
and the $\phi^a_c$ as defined in (2) (3.19) are indeterminate.

Further, for a $C$-type system,
$$l^b C_b{}^a{}_c = 0, \qquad \theta^a_c = \delta^a_c, \qquad \theta = 1,$$
whereas, in the case now being considered, $\theta = 0$. Hence, since the $A_b{}^a{}_c$ and $\Gamma^*{}_b{}^a{}_c$ are functions of the $\phi^a_c$, from (2) (3.20):

*If the element of support of an F–R system is transported along trajectories by induced parallelism in $S_n$ with deformation of $S_n$, then* (i) $A_b{}^a{}_c$ *and* $\Gamma^*{}_b{}^a{}_c$ *are indeterminate, and* (ii) *the system cannot be C-type, and therefore cannot be complete.*

Since $Dl^a = B^a_\alpha Dl^\alpha$, the same result follows if the element of support is transported along trajectories by parallelism in $S_\nu$.

## 6. The function $\gamma_{abc}$

For the Finsler image we define, as in (1) (15),
$$\gamma|_{abc} = \frac{1}{2}\left[\frac{\partial g|_{bc}}{\partial x^a} + \frac{\partial g|_{ab}}{\partial x^c} - \frac{\partial g|_{ac}}{\partial x^b}\right]. \tag{6.1}$$

Similarly, for the F–R system, we define
$$\gamma_{abc} = \frac{1}{2}\left[\left(\frac{\partial g_{bc}}{\partial x^a}\right)_u + \left(\frac{\partial g_{ab}}{\partial x^c}\right)_u - \left(\frac{\partial g_{ac}}{\partial x^b}\right)_u\right]. \tag{6.2}$$

Now
$$\left(\frac{\partial g_{bc}}{\partial x^a}\right)_u = \left(\frac{\partial g_{bc}}{\partial x^a}\right)_t + \frac{\partial g_{bc}}{\partial t^A} p^A_c, \quad \text{etc.,}$$
where the $\left(\frac{\partial g_{bc}}{\partial x^a}\right)_u$, $\left(\frac{\partial g_{bc}}{\partial x^a}\right)_t$ are defined as in (2) § 4.

Therefore
$$\gamma_{abc} = [ac, b] + \frac{1}{2}\left(\frac{\partial g_{bc}}{\partial t^A} p^A_a + \frac{\partial g_{ab}}{\partial t^A} p^A_c - \frac{\partial g_{ac}}{\partial t^A} p^A_b\right). \tag{6.3}$$

## 7. Existence theorems

Given a function $2F(x,u)$, homogeneous of second order in the $u^a$, I shall consider whether a complete F–R system exists in which $L^2 = 2F$.

**7.1.** In order that the system shall be normal, it is necessary and sufficient that $g_{ab} = \partial^2 F/\partial u^a \partial u^b$, and, if simplifying $S_\nu$-coordinates are used as in § 3, we shall have $g'_{ab} = g_{ab}$. Hence, *a necessary and sufficient condition for the system to be normal is*

$$g'_{ab} = \partial^2 F/\partial u^a \partial u^b. \tag{7.1}$$

This expression is homogeneous of zero order in the $u^a$, and, after substitution for the ratios $u^1/u^2/.../u^n$ from the equations (**2**) (2.5)

$$u^a/u^n = f^a(x,t)/f^n(x,t) \quad (a = 1, 2,..., n-1), \tag{7.2}$$

which will result from our choice of generators and which have yet to be determined, the $g'_{ab}$ become functions of the $x^a$ and $t^A$, and finally, by means of (3.1), functions of the $y^\alpha$.

**7.2.** In order that this normal system shall also be $C$-type, from the second result of § 4 it is necessary and sufficient that $C_{abc} = C_{bac}$. Using (3.6) we can write this condition

$$([aA,b]'-[bA,a]')q_c^A = 0,$$

i.e.
$$\left(\frac{\partial g'_{aA}}{\partial y^b} - \frac{\partial g'_{bA}}{\partial y^a}\right) q_c^A = 0. \tag{7.3}$$

Hence, *a sufficient condition that the normal system shall be $C$-type is that*

$$\frac{\partial g'_{aA}}{\partial y^b} = \frac{\partial g'_{bA}}{\partial y^a}. \tag{7.4}$$

These equations can always be satisfied by taking the $g'_{aA} = 0$ (when the trajectories will be orthogonal as in (**2**) § 5), by taking $g'_{bA}$ to be independent of all the $y^\alpha$ except $y^b$, or otherwise.

In general, the condition (7.4) is not *necessary* for the normal system to be $C$-type, but in the case in which $n = 2$ equations (7.3) reduce to

$$\left(\frac{\partial g'_{13}}{\partial y^2} - \frac{\partial g'_{23}}{\partial y^1}\right) q_c^3 = 0,$$

and, since $q_1^3$ and $q_2^3$ cannot vanish ($t^3$ being supposed not independent of the $u^a$), we must have

$$\frac{\partial g'_{13}}{\partial y^2} = \frac{\partial g'_{23}}{\partial y^1}$$

as a necessary condition for the normal system to be $C$-type when $n = 2$.

**7.3.** Returning to the general case, to obtain the condition that this normal, $C$-type system shall also be $\Gamma$-type, we evaluate

$$D_b{}^a{}_c = \Gamma_b{}^a{}_c - \Gamma|_b{}^a{}_c. \tag{7.5}$$

From (3.3), (3.6),

$$D_b{}^a{}_c = g^{ae}([bc,e]+[bA,e]'p_c^A) - \Gamma|_b{}^a{}_c,$$

whence

$$D_{bfc} = [bc,f]+[bA,f]'p_c^A - \Gamma|_{bfc}. \tag{7.6}$$

From (7.4),

$$[bA,f]' = \frac{1}{2}\left(\frac{\partial g'_{bf}}{\partial y^A}+\frac{\partial g'_{fA}}{\partial y^b}-\frac{\partial g'_{bA}}{\partial y^f}\right) = \frac{1}{2}\frac{\partial g'_{bf}}{\partial y^A}.$$

Using this result and (6.3) in (7.6) we have

$$D_{bfc} = \gamma_{bfc} - \frac{1}{2}\left(\frac{\partial g_{cf}}{\partial t^A}p_b^A + \frac{\partial g_{bf}}{\partial t^A}p_c^A - \frac{\partial g_{bc}}{\partial t^A}p_f^A\right) + \frac{1}{2}\frac{\partial g'_{bf}}{\partial y^A}p_c^A - \Gamma|_{bfc}.$$

But

$$\frac{\partial g'_{bf}}{\partial y^A} = \frac{\partial g_{bf}}{\partial t^A}$$

since $g'_{bf} = g_{bf}$ and $y^A = t^A$; also $\gamma_{bfc} = \gamma|_{bfc}$ since $g_{ab} = g|_{ab}$. Then

$$D_{bfc} = \frac{1}{2}\left(\frac{\partial g_{bc}}{\partial t^A}p_f^A - \frac{\partial g_{cf}}{\partial t^A}p_b^A\right) - S|_{bfc}, \tag{7.7}$$

where

$$S|_{bfc} = \Gamma|_{bfc} - \gamma|_{bfc}. \tag{7.8}$$

The value of $S|_{bfc}$ is given by (**1**) 16, viz.

$$S|_{bfc} = C|_{bce}\frac{\partial G|^e}{\partial u^f} - C|_{fce}\frac{\partial G|^e}{\partial u^b}, \tag{7.9}$$

where

$$2G|_m = \frac{\partial^2 F}{\partial u^m \partial x^a}u^a - \frac{\partial F}{\partial x^m}. \tag{7.10}$$

Also, from (7.7),

$$u^c D_{bfc} = \tfrac{1}{2}u^c\left(\frac{\partial g_{bc}}{\partial t^A}p_f^A - \frac{\partial g_{cf}}{\partial t^A}p_b^A\right) \tag{7.11}$$

since, from (**1**) 15,

$$u^c S|_{bfc} = 0.$$

Now

$$\frac{\partial g_{bc}}{\partial t^A}q_f^A = \frac{\partial g_{bc}}{\partial u^f} = 2C_{bcf}, \tag{7.12}$$

and for given values of $b$, $c$ we can solve any $n-1$ of these equations for the $n-1$ unknowns $\dfrac{\partial g_{bc}}{\partial t^A}$. Thus, if $q_A^{[f]}$ is the minor of $q_{[f]}^A$ in $|q_{[f]}^A|$ divided by this determinant, where the brackets [ ] enclosing an affix indicate that it is to run from 1 to $n-1$ only, then

$$q_{[f]}^A q_B^{[f]} = \delta_B^A. \tag{7.13}$$

The equations (7.12) now give

$$\frac{\partial g_{bc}}{\partial t^A} q^A_{[f]} q^{[f]}_B = 2C_{bc[f]} q^{[f]}_B,$$

i.e.
$$\frac{\partial g_{bc}}{\partial t^B} = 2C_{bc[f]} q^{[f]}_B. \tag{7.14}$$

Hence
$$u^b \frac{\partial g_{bc}}{\partial t^B} = 0 \tag{7.15}$$

since $u^b C_{bcf} = 0$ for a normal, $C$-type system (as proved in § 4). Then, from (7.11), $u^c D_{bfc} = 0$,

i.e.
$$u^c \Gamma_{bfc} = u^c \Gamma|_{bfc},$$

and
$$u^c \Gamma_b{}^a{}_c = u^c \Gamma|_b{}^a{}_c. \tag{7.16}$$

Thus
$$dx^c \Gamma_b{}^a{}_c = dx^c \Gamma|_b{}^a{}_c$$

if the $dx^c$ are proportional to the $u^c$, and

$$DX^a = DX|^a \tag{7.17}$$

*for any displacement in the direction of the element of support* for the system defined by (7.1) and (7.4). Thus:

*Given the function $2F(x, u)$, a system in which*

(i) $g'_{ab} = \dfrac{\partial^2 F}{\partial u^a \partial u^b}$,

(ii) *the* $g'_{aA}$ *satisfy* $\dfrac{\partial g'_{bA}}{\partial y^a} = \dfrac{\partial g'_{aA}}{\partial y^b}$

*is a normal C-type system having $L^2$ equal to $2F(x, u)$, and, although it is not, in general, complete, $DX^a = DX|^a$ for displacement in the direction of the element of support. The $g'_{AB}$ and the generators are arbitrary.*

Hence:

THEOREM I. *Given a function $2F(x, u)$, homogeneous of second order in the $u^a$, there exist normal, C-type systems in which $L^2 = 2F$.*

From (7.9), $S|_{bac} = -S|_{abc}$ and then, from (7.7), $D_{bac} = -D_{abc}$, i.e. $D_{bac} = 0$ when $b = a$. From (7.7) it follows that a necessary and sufficient condition that the normal, $C$-type system defined by (7.1) and (7.4) shall be $\Gamma$-type (and hence complete) is that

$$\frac{1}{2}\left(\frac{\partial g_{bc}}{\partial t^A} p^A_a - \frac{\partial g_{ac}}{\partial t^A} p^A_b\right) = S|_{bac} \quad \text{when } b \neq a. \tag{7.18}$$

**7.4.** In the special case in which $F$ is independent of the $x^a$, we have $\gamma|_{abc} = 0$, $G|_m = 0$ from (7.10), and $S|_{bac} = 0$ from (7.9); hence

$\Gamma|_{abc} = 0$ from (7.8). In this case (7.18) can be satisfied simply by choosing generators in such a way that

$$p_a^A = 0; \tag{7.19}$$

this can always be done by defining generators by equations (**2**) (7.2) (or otherwise). If (**2**) (7.2) are now solved for the $t^A$, the values obtained are independent of the $x^a$ and these equations and the ratios $u^1/u^2/.../u^n$ must therefore be independent of the $x^a$. Since

$$l^a = u^a/L = u^a/\sqrt{(g_{bc}\,u^b u^c)},$$

the $l^a$ are homogeneous of zero order in the $u^a$ and through (7.2) they can be expressed in terms of the $t^A$ independently of the $x^a$ since the $g_{ab}$ are now also independent of the $x^a$; the components of the unit vector in the direction of the element of support are thus the same at all points of $S_n$ ($t^A$ = constant), and the $l$-vector at any point of $S_n$ is therefore parallel with respect to $S_n$ to the $l$-vector at any other point of $S_n$. The generators are thus extremals of $S_n$, parallel to each other with respect to $S_n$. Thus:

*Given the function $2F(u)$, independent of the $x^a$, a system in which*

(i) $g'_{ab} = \dfrac{\partial^2 F}{\partial u^a \partial u^b}$,

(ii) *the $g'_{aA}$ satisfy* $\dfrac{\partial g'_{bA}}{\partial y^a} = \dfrac{\partial g'_{aA}}{\partial y^b}$,

(iii) *the generators are extremals of $S_n$ parallel to each other with respect to $S_n$*

*is a complete system having $L^2$ equal to $2F(u)$. The $g'_{AB}$ are arbitrary.*

If $2F(x,u)$ is not independent of the $x^a$ but can be made so by a transformation of $S_n$-coordinates from $x^a$ to $x''^a$, i.e. $2F(x,u) = 2F''(u'')$ say, a complete system can be determined as above for which

$$L^2 = 2F''(u''),$$

and, on returning to the original coordinates, we get $L^2 = F$, and the system remains complete by the result of § 2. Hence

THEOREM II. *Given a function $2F$, homogeneous of second order in the $u^a$, which either is independent of the $x^a$ or can be made so by a transformation of $S_n$-coordinates, there exist complete systems in which $L^2 = 2F$.*

**7.5.** Returning to the general case in which $2F$ is not independent of the $x^a$ (and cannot be made so by a change of $S_n$-coordinates), we can write equations (7.18)

$$C_{bc[h]}\,q_A^{[h]}\,p_f^A - C_{cf[h]}\,q_A^{[h]}\,p_b^A = S|_{bfc}$$

using (7.14); these are equations in the $p_a^A$ and $q_a^A$, i.e. $\partial t^A/\partial x^a$ and $\partial t^A/\partial u^a$, which could be used to determine the $t^A$ in terms of the $x^a$ and $u^a$, and hence to obtain the generators. In the general case, however, these equations appear intractable, and I shall consider only the case in which $n = 2$. Equations (7.18) now reduce to

$$\frac{1}{2}\left(\frac{\partial g_{1c}}{\partial t^3}p_2^3 - \frac{\partial g_{2c}}{\partial t^3}p_1^3\right) = S|_{12c}, \qquad (7.20)$$

and equations (7.12) reduce to

$$\frac{\partial g_{ab}}{\partial t^3}q_c^3 = \frac{\partial g_{ab}}{\partial u^c} = 2C_{abc}, \qquad (7.21)$$

whence

$$\frac{1}{2}\frac{\partial g_{ab}}{\partial t^3} = C_{ab(c)}/q_{(c)}^3, \qquad (7.22)$$

where the brackets ( ) enclosing an affix indicate that we do not sum with respect to that affix. The right-hand side of (7.22) is independent of the bracketed affixes since $u^c C_{abc} = 0$ and $u^c q_c^3 = 0$ from (**2**) (3.16); whence

$$C_{ab1}/q_1^3 = C_{ab2}/q_2^3.$$

Equations (7.20) can now be written

$$p_2^3 C_{1c1}/q_1^3 - p_1^3 C_{2c1}/q_1^3 = S|_{12c}.$$

Since $u^c S|_{12c} = 0$ from (**1**) 15, the two equations obtained by writing $c = 1, 2$ are not independent, and they are equivalent to the single equation (we take $c = 1$)

$$C_{111}p_2^3 - C_{211}p_1^3 = S|_{121}q_1^3.$$

Then, since $u^1C_{111}+u^2C_{211} = 0$, we have

$$u^1p_1^3+u^2p_2^3 = (S|_{121}/C_{111})u^2q_1^3. \qquad (7.23)$$

Now $\quad q_1^3 = \partial t^3/\partial u^1 = (\partial t^3/\partial r)(\partial r/\partial u^1) = (\partial t^3/\partial r)/u^2,$

where $r = u^1/u^2$, and the right-hand side of (7.23) becomes

$$(S|_{121}/C_{111})(\partial t^3/\partial r).$$

Dividing (7.23) by $u^2$ we get

$$r\,\partial t^3/\partial x^1+\partial t^3/\partial x^2 = (S|_{121}/u^2C_{111})(\partial t^3/\partial r), \qquad (7.24)$$

in which $S|_{121}/u^2C_{111}$, being of zero order in the $u^a$, can be expressed in terms of $r$. When we write

$$S|_{121}/u^2C_{111} = H(x^1, x^2, r), \qquad (7.25)$$

(7.24) becomes

$$r\,\partial t^3/\partial x^1+\partial t^3/\partial x^2 = H(x^1, x^2, r)\,\partial t^3/\partial r. \qquad (7.26)$$

Solving (7.26) enables us to express $t^3$ in terms of $x^1$, $x^2$, $r$ and hence to express $r$ in terms of $x^1$, $x^2$, $t^3$ in the form $r = r(x^1, x^2, t^3)$. The generators (for which $dx^1/dx^2 = u^1/u^2 = r$) are then obtained by solving the differential equation

$$dx^1/dx^2 = r(x^1, x^2, t^3) \quad (t^3 = \text{constant}).$$

Thus

*Given the function $2F(x, u)$, and $n = 2$, a system in which*

(i) $g'_{ab} = \dfrac{\partial^2 F}{\partial u^a \partial u^b}$,

(ii) *the $g'_{b3}$ satisfy* $\dfrac{\partial g'_{b3}}{\partial y^c} = \dfrac{\partial g'_{c3}}{\partial y^b}$,

(iii) *the generators are obtained by solving*

$$r\,\partial t^3/\partial x^1 + \partial t^3/\partial x^2 = H(x^1, x^2, r)\,\partial t^3/\partial r$$

*is a complete system having $L^2$ equal to $2F(x, u)$. The coefficient $g'_{33}$ is arbitrary.*

Hence we are led to:

Theorem III. *If $n = 2$, given a function $2F(x, u)$, homogeneous of second order in the $u^a$, there exist complete systems in which $L^2 = 2F$, provided that the equation*

$$r\,\partial t^3/\partial x^1 + \partial t^3/\partial x^2 = H(x^1, x^2, r)\,\partial t^3/\partial r$$

*(which gives the generators) has a solution which is valid for all values of $x^1$, $x^2$, $r$.*

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# THE TOTAL LENGTH OF THE EDGES OF A POLYHEDRON

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IT has been conjectured by L. Fejes Tóth (**2**) that the edges of any convex polyhedron which contains a sphere of unit radius have a total length of at least 24. Our object is to establish the truth of this conjecture. The minimal length 24 is attained for a cube and only for a cube. Partial results have been obtained by Hammersley (**1**) and by Fejes Tóth (**2**, **3**), who established his conjecture under the additional restriction that the faces of the polyhedron are of equal area, a restriction which alters the whole nature of the problem.

For any convex polyhedron $P$ we denote by $L(P)$ the total length of its edges and by $S_P$ its insphere, or one of its inspheres if there is more than one. The radius of $S_P$ is denoted by $R(P)$. Let $\mathscr{P}(N)$ be the class of convex polyhedra with at most $N$ faces. We shall assume in what follows that $N \geqslant 6$, so that $\mathscr{P}(N)$ contains all cubes.

Write $\qquad L_N^* = \inf L(P), \qquad P \in \mathscr{P}(N), \qquad R(P) = 1.$

There is at least one member of $\mathscr{P}(N)$, say $P_N^*$, for which

$$L(P_N^*) = L_N^*, \qquad R(P_N^*) = 1.$$

Since $N \geqslant 6$, we know that $L_N^* \leqslant 24$.

In what follows we suppose that $N$ is fixed and, for convenience of notation, use $P$, $L$, $S$ in place of $P_N^*$, $L(P_N^*)$, $S_{P_N^*}$ respectively.

The method is to establish a sequence of properties of $P$ which finally imply that it is a cube. These properties are numbered (i) to (xi), and the proofs of some of them, which are of an elementary nature, are placed in an Appendix.

Denote the vertices of $P$ by $D(1), D(2),..., D(k)$ and the planes which contain the faces of $P$ by $\pi(1), \pi(2),..., \pi(m)$. The face of $P$ in $\pi(i)$, i.e. the closed point set $P \cap \pi(i)$, is denoted by $F(i)$. The centre of $S$ is denoted by $O$.

(i) *We may assume that every face of $P$ touches $S$.*

If there is a face of $P$ that does not touch $S$, let the notation be so chosen that this face is $\pi(1)$. Let the distance of $\pi(1)$ from $O$ be $x_1$ ($> 1$) and let $\tau(x)$ be the plane parallel to $\pi(1)$ at a distance $x$ from $O$ such

that $O$ does not lie between $\pi(1)$ and $\tau(x)$. Denote by $P(x)$ the polyhedron that is bounded by $\tau(x)$, $\pi(2),\ldots, \pi(m)$.

We show next that $L\{P(x)\}$ is a constant function of $x$ in a neighbourhood of $x_1$. To do this it is convenient to define a new polyhedron $Q$ which is that polyhedron bounded by $\pi(1)$ and all those other $\pi(i)$ which either do not meet $F(1)$ or alternatively do so in a non-degenerate segment. Suppose that $Q$ is bounded by $\pi(1)$, $\pi(i_1),\ldots, \pi(i_n)$. Let $Q(x)$ be the polyhedron bounded by $\tau(x)$, $\pi(i_1),\ldots, \pi(i_n)$.

To obtain estimates of $L(P)-L\{P(x)\}$ and $L(Q)-L\{Q(x)\}$ we need the following lemma.

Lemma. *Let $A_0, A_1,\ldots, A_s$ be a convex polygon and $B_0$ a point, all lying in one plane $\pi$ such that $A_1, A_2,\ldots, A_{s-1}$ lie inside the triangle $A_0 A_s B_0$. Let $D$ be a point not in $\pi$. Then*

$$A_0 B_0+A_s B_0-DB_0+\sum_{i=0}^{s} DA_i-\sum_{i=1}^{s} A_{i-1} A_i > 0, \tag{1}$$

*where $A_0 B_0$ is used for the length of the segment joining $A_0$ to $B_0$, etc.*

Produce $A_i A_{i+1}$ to meet $A_s B_0$ in $B_{i+1}$ $(i = 1,\ldots, s-1)$. Then $A_s = B_s$, and the points $B_0, B_1,\ldots, B_s$ are in order on $B_0 A_s$. Since the sum of the lengths of two pairs of opposite sides of a tetrahedron is greater than the sum of the lengths of the remaining pair, we obtain from $DB_{i+1} B_i A_i$ the inequality

$$(DB_{i+1}-DB_i)+DA_i+B_i B_{i+1}+(A_i B_i-A_i B_{i+1}) > 0 \quad (i = 0,\ldots, s-1).$$

If we add all these inequalities together, we obtain (1).

Now, if $x > x_1$, then $P(x) \supset P$, $Q(x) \supset Q$, the set of points in $P(x)$ and not in $P$ is identical with the set in $Q(x)$ and not in $Q$, and the face of $Q$ in $\pi(1)$ coincides with $F(1)$. Thus

$$L\{P(x)\}-L(P) = L\{Q(x)\}-L(Q). \tag{2}$$

If $x < x_1$, then suppose that $F(1)$, the face of $P$ in $\pi(1)$, has vertices $D(1),\ldots, D(t)$ (see Fig. 1 which illustrates the case $t = 5$). Let the edges of $P$ through $D(i)$, other than those in $\pi(1)$, meet $\tau(x)$ in

$$A_i(0),\ A_i(1),\ldots,\ A_i(l)$$

and let the totality of these points be in order on the frontier of $P \cap \pi(x)$. The integer $l$ is not fixed but takes different values for different values of $i$. Similarly let the edge of $Q$ that passes through $D(i)$ and does not lie in $\pi(1)$ meet $\tau(x)$ in $B_i$. If there is only one edge of $P$ through $D(i)$, other than those in $\pi(1)$, then $A_i(0)$ coincides with $B_i$.

For $x$ near to $x_1$ ($x < x_1$) we have

$$L\{Q(x)\}-L(Q)-L\{P(x)\}+L(P)$$
$$=\sum_{i=1}^{t}\Big\{A_i(0)B_i+B_i A_i(l)-D(i)B_i+\sum_{j=0}^{l} D(i)A_i(j)-\sum_{j=1}^{l} A_i(j-1)A_i(j)\Big\}, \tag{3}$$

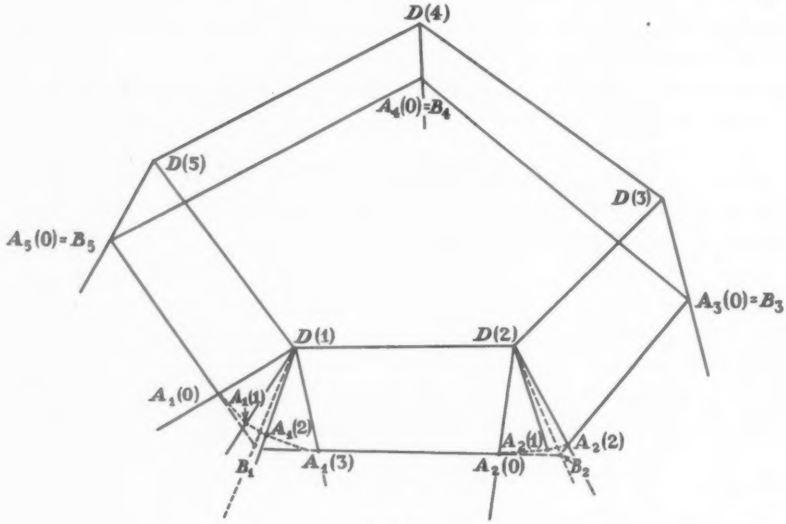


FIG. 1.

where the expression in braces is understood to be zero when $A_i(0)$ is $B_i$.

From the lemma and (3) it follows that

$$L\{P(x)\}-L(P) \leqslant L\{Q(x)\}-L(Q) \quad (x \leqslant x_1). \tag{4}$$

But $P(x) \in \mathscr{P}(N)$ and $R\{P(x)\} = 1$ if $x$ is sufficiently close to $x_1$. Thus, by the extremal property of $P$,

$$L\{P(x)\} \geqslant L(P), \tag{5}$$

for all $x$ near to $x_1$. However, $L\{Q(x)\}$ is a linear function of $x$, and this fact combined with (2), (4), and (5) implies that it is a constant. But then, unless $P$ is identical with $Q$, we have strict inequality in (4). By (5) this is impossible. Thus $P$ is $Q$ and $L\{P(x)\}$ is constant as $x$ varies. We replace $P$ by $P(x)$, where $x = R(P) = 1$. The values of $L(P)$ and $R(P)$ remain unchanged.

This establishes (i). In what follows we assume that every face of $P$ touches $S$. We return to a consideration of this assumption at the end of this paper.

(ii) *Every vertex of $P$ lies on exactly three edges of $P$.*

We use the same notation as in (i) and assume that $D(1), D(2),..., D(t)$ are the vertices of $P$ in $\pi(1)$ and that $\pi(1)$ meets $S$ in the point $M$. We shall show that the statement (ii) is true of all the vertices of $P$ in $\pi(1)$. Since $\pi(1)$ is any one of the bounding planes of $P$, the result is then true generally.

Denote by $P_1$ the solid which contains $P$ and whose frontier lies in the planes $\pi(2),..., \pi(m)$. This solid may be unbounded. Let $\pi(1,\theta)$ be the plane obtained from $\pi(1)$ by rotating through an angle $\theta$ about an axis of rotation which lies in $\pi(1)$ and passes through $M$. We choose this axis of rotation so that it does not pass through any vertex of $P$. $P_1$ is divided by $\pi(1,\theta)$ into two components: that which contains $O$ is denoted by $P_1(\theta)$. We are interested only in the case when $\theta$ is a small angle.

We have
$$R\{P_1(\theta)\} = R(P)+O(\theta^2) \quad (\theta \to 0) \tag{6}$$
and, since $P_1(\theta) \in \mathscr{P}(N)$,
$$\frac{L(P)}{R(P)} \leqslant \frac{L\{P_1(\theta)\}}{R\{P_1(\theta)\}}. \tag{7}$$
Thus
$$L\{P_1(\theta)\} \geqslant L(P)-\eta, \qquad \eta = O(\theta^2). \tag{8}$$

Let $Q$ be the polyhedron defined in (i) and let $Q_1(\theta)$ be obtained from $Q$ in the same way that $P_1(\theta)$ was obtained from $P$, i.e. by replacing $\pi(1)$ by $\pi(1,\theta)$.

Of the vertices of $P$ in $\pi(1)$ suppose that $D(1),..., D(s)$ lie on one side of the axis of rotation and $D(s+1),..., D(t)$ lie on the other side. Let the vertices of $P_1(\theta)$ on the plane $\pi(1,\theta)$ lying on edges of $P$ which terminate at $D(i)$ be $A(i,1),..., A(i,l)$ and let $B(i)$ be the vertex of $Q_1(\theta)$ that lies on the edge of $Q$ terminating at $D(i)$. The points $A(i,j)$, $B(i)$ are not vertices of $P$ or $Q$ respectively. (See Fig. 2 for an illustration of the case $t = 5$.)

Let the sign of $\theta$ be such that, for $\theta \geqslant 0$, $A(i,j) \in P$ for $1 \leqslant i \leqslant s$. Then

$$\begin{aligned} &L\{Q_1(\theta)\}-L(Q)-L\{P_1(\theta)\}+L(P) \\ &\quad = \sum \Big\{\sum_{j=1}^{l} D(i)A(i,j)-\sum_{j=1}^{l-1} A(i,j)A(i,j+1)-D(i)B(i)+ \\ &\qquad\qquad +A(i,1)B(i)+A(i,l)B(i)\Big\}, \end{aligned} \tag{9}$$

where the unspecified range of summation is from $i = 1$ to $i = t$. The expression inside the braces is to be replaced by zero if $s+1 \leqslant i \leqslant t$ and $\theta \geqslant 0$ or if $1 \leqslant i \leqslant s$ and $\theta \leqslant 0$.

Now let the plane through $B(i)$ parallel to $\pi(1)$ meet $D(i)A(i,j)$, pro-

duced if necessary, in $A'(i,j)$. Then the distance $A(i,j)A'(i,j) = O(\theta^2)$. Further, if we consider different values of $\theta$, all of the same sign, the resulting figures involving $D(i)B(i)A'(i,j)$ $(j = 1,..., l)$ are all similar. Hence

$$\sum_{j=1}^{l} D(i)A'(i,j) - \sum_{j=1}^{l-1} A'(i,j)A'(i,j+1) - D(i)B(i) +$$
$$+ A'(i,1)B(i) + A'(i,l)B(i) = \lambda(i)D(i)B(i), \quad (10)$$

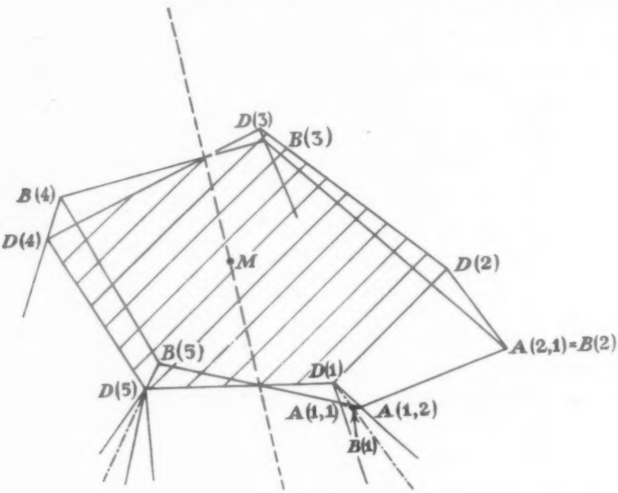


FIG. 2.

where $\lambda(i)$ is a constant independent of $\theta$, provided that $\theta$ is of one sign. Now, by the lemma proved in (i),

$$\lambda(i) \geqslant 0 \quad (1 \leqslant i \leqslant t),$$

and further $\lambda(i) > 0$ for every vertex $D(i)$ that lies on more than three edges of $P$ and for which $1 \leqslant i \leqslant s$ when $\theta \geqslant 0$. Also

$$D(i)B(i) = \mu(i)\theta + O(\theta^2), \qquad \mu(i) > 0 \quad (1 \leqslant i \leqslant s;\ \theta \geqslant 0). \quad (11)$$

From (9), (10), (11) we obtain for $\theta$ small and positive

$$L\{Q_1(\theta)\} - L(Q) - L\{P_1(\theta)\} + L(P) = \sum_{i=1}^{s} \lambda(i)\mu(i)\theta + O(\theta^2). \quad (12)$$

A similar expression holds when $\theta$ is negative. Also $L\{Q_1(\theta)\} - L(Q)$ is a differentiable function of $\theta$ at $\theta = 0$. Thus

$$L\{Q_1(\theta)\} - L(Q) = A\theta + o(\theta),$$

where $A$ is a fixed constant and $\theta$ may be positive or negative. Hence, from (12),

$$L\{P_1(\theta)\}-L(P) = A\theta-f(\theta)+o(\theta), \tag{13}$$

where

$$f(\theta) = \begin{cases} K_1\theta & (\theta \geqslant 0), \\ -K_2\theta & (\theta \leqslant 0) \end{cases}$$

and $K_1$, $K_2$ are non-negative constants.

Now $$L\{P_1(\theta)\} \geqslant L(P)-\eta, \qquad \eta = O(\theta^2).$$

Thus $$A\theta \geqslant f(\theta),$$

i.e. $$A-K_1 \geqslant 0, \qquad A+K_2 \leqslant 0. \tag{14}$$

Hence $K_1 = K_2 = A = 0$. This implies that $P$ and $Q$ coincide.

Thus every vertex of $P$ in $\pi(1)$ lies on exactly three edges and the same is true for all the vertices of $P$.

It is convenient now to introduce some new notation. We shall use $D$ to denote any one of the vertices of $P$, and we shall denote the three faces of $P$ that meet at $D$ by $F(1)$, $F(2)$, $F(3)$. The planes containing these faces are denoted by $\pi(1)$, $\pi(2)$, $\pi(3)$ respectively. Let the point of contact of $\pi(i)$ with $S$ be $M_i$ and the point of intersection of the three planes $\pi(i)$, $\pi(j)$, and $OM_i M_j$ be $J_k$, where $i, j, k$ is a permutation of 1, 2, 3. Denote the spherical triangle $M_1 M_2 M_3$ by $T(D)$ and its area by $A(D)$. Let the angle of $T(D)$ at $M_i$ be $\alpha_i$. Let the sum of the lengths $DJ_1+DJ_2+DJ_3$ be $l(D)$, where $DJ_1$ is positive if $D$ and $M_1$ lie on the same side of the plane $OM_2 M_3$ and otherwise $DJ_1$ is negative; there are similar sign conventions for $DJ_2$ and $DJ_3$.

It will be seen that the spherical triangles $T(D)$ for different vertices $D$ do not overlap and that their totality covers the surface of $S$. Further $\sum l(D) = L(P)$, where the sum is over all the vertices $D$.

(iii) *$l(D)$ is given by the formula*

$$\frac{l(D)}{\{\sin\frac{1}{2}A(D)\}^{\frac{1}{2}}} = \frac{\cos\{\alpha_1-\frac{1}{2}A(D)\}+\cos\{\alpha_2-\frac{1}{2}A(D)\}+\cos\{\alpha_3-\frac{1}{2}A(D)\}}{[\sin\{\alpha_1-\frac{1}{2}A(D)\}\sin\{\alpha_2-\frac{1}{2}A(D)\}\sin\{\alpha_3-\frac{1}{2}A(D)\}]^{\frac{1}{2}}}. \tag{15}$$

Let $\pi'(i)$ denote the plane which is obtained from $\pi(i)$ by reflection in $O$. Define the following points as intersections of three planes (see Fig. 3).

$$E = \pi(1) \cap \pi'(2) \cap \pi'(3), \qquad E' = \pi'(1) \cap \pi(2) \cap \pi(3),$$
$$F = \pi(1) \cap \pi'(2) \cap \pi(3), \qquad F' = \pi'(1) \cap \pi(2) \cap \pi'(3),$$
$$G = \pi'(1) \cap \pi'(2) \cap \pi(3), \qquad G' = \pi(1) \cap \pi(2) \cap \pi'(3),$$
$$D' = \pi'(1) \cap \pi'(2) \cap \pi'(3).$$

Let the intersection of the lines $M_3 J_1$ and $FG$ be $U_1$ and of the lines $M_3 J_2$ and $E'G$ be $U_2$. Further let $M_2 J_1$ meet $G'F'$ in $V_1$. Since

$$M_2 J_1 = M_3 J_1 = \tan \tfrac{1}{2}\angle M_2 O M_3, \qquad M_3 U_1 = M_2 V_1 = \cot \tfrac{1}{2}\angle M_2 O M_3,$$

the width of the parallelogram $DE'GF$ perpendicular to $DE'$ is equal to the width of the parallelogram $DE'F'G'$ perpendicular to $DE'$. Thus

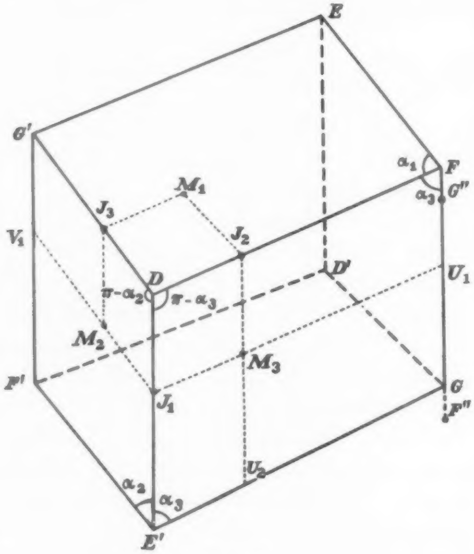


FIG. 3.

on the line $FG$ there are two points $G''$ and $F''$ such that the parallelogram $G''DE'F''$ is congruent to the parallelogram $G'DE'F'$.

If $S$ is projected from $D'$ on to $\pi(3)$, the frontier of the projected set is a parabola $\Lambda$ of which $M_3$ is the focus and $FG$, $GE'$ are two tangents. Further a similar projection of $S$ from $D'$ on to $\pi(2)$ leads to a parabola $\Lambda_1$ in $\pi(2)$. Now $D'$ and $O$ both lie on the plane bisecting the angle between the planes $\pi(2)$ and $\pi(3)$; thus $\Lambda_1$ is congruent to $\Lambda$. Since $E'F'$ is a tangent to $\Lambda_1$, $E'F''$ is a tangent to $\Lambda$. But then, from a well-known property of the parabola, $M_3$, $G$, $F''$, $E'$ are four concyclic points. Then either $E'G$ separates $M_3$ from $F''$ in $\pi(3)$ and $\angle GM_3E' = \pi - \angle GF''E' = \alpha_2$ or $M_3$, $F''$ lie on the same side of $E'G$ and $\angle E'F''G = \angle GM_3E' = \alpha_2$. Similarly $\angle DM_3E' = \pi - \alpha_1$.

Now, in the plane $\pi(3)$, $J_2 M_3 U_2$ is perpendicular to $DF$ and to $E'G$, whilst $J_1 M_3 U_1$ is perpendicular to $DE'$ and to $FG$. Thus the sets of

points $D$, $J_2$, $M_3$, $J_1$ and $M_3$, $U_2$, $G$, $U_1$ are concyclic. Also, since

$$J_1 M_3 . M_3 U_1 = 1, \qquad J_2 M_3 . M_3 U_2 = 1,$$

we have $J_1 M_3 . M_3 U_1 = J_2 M_3 . M_3 U_2$ and $J_1 J_2 U_2 U_1$ are concyclic. Hence

$$\angle E'DM_3 = \angle J_1 J_2 M_3 = \angle J_1 U_1 U_2 = \angle M_3 GE'.$$

Thus from the triangles $DM_3 E'$, $GM_3 E'$ we obtain

$$\angle E'DM_2 = \tfrac{1}{2}(\pi+\alpha_1-\alpha_2-\alpha_3) = \alpha_1-\tfrac{1}{2}A(D).$$

Similarly

$$\angle FDM_3 = \alpha_2-\tfrac{1}{2}A(D), \qquad \angle G'DM_2 = \alpha_3-\tfrac{1}{2}A(D).$$

Now

$$l(D) = \frac{J_1 M_3}{\sin \angle E'DM_3}\{\cos \angle FDM_3+\cos \angle E'DM_3+\cos \angle G'DM_2\}. \quad (16)$$

But
$$J_1 M_3 = \tan \tfrac{1}{2} \angle M_2 OM_3$$

and
$$\cos \angle M_2 OM_3 = \frac{\cos\alpha_1+\cos\alpha_2 \cos\alpha_3}{\sin\alpha_2 \sin\alpha_3}.$$

Thus

$$\begin{aligned}(J_1 M_3)^2 &= \frac{\sin\alpha_2 \sin\alpha_3-\cos\alpha_2 \cos\alpha_3-\cos\alpha_1}{\sin\alpha_2 \sin\alpha_3+\cos\alpha_2 \cos\alpha_3+\cos\alpha_1} \\ &= -\frac{\cos(\alpha_2+\alpha_3)+\cos\alpha_1}{\cos(\alpha_2-\alpha_3)+\cos\alpha_1} \\ &= -\frac{\cos\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3)\cos\frac{1}{2}(\alpha_1-\alpha_2-\alpha_3)}{\cos\frac{1}{2}(\alpha_1+\alpha_2-\alpha_3)\cos\frac{1}{2}(-\alpha_1+\alpha_2-\alpha_3)} \\ &= \frac{\sin\frac{1}{2}A \sin(\alpha_1-\frac{1}{2}A)}{\sin(\alpha_3-\frac{1}{2}A)\sin(\alpha_2-\frac{1}{2}A)},\end{aligned}$$

where we have written $A$ for $A(D)$.

Substitution for $J_1 M_3$ in (16) leads to the formula (15). Note that there is no ambiguity of sign in taking the square root since $J_1 M_3$ is always a positive distance.

Although we introduced the triangles $T(D)$ as functions of $D$, it is also convenient to adopt the inverse point of view and to regard $D$ as obtained from the triangle $T(D)$ by forming the point of intersection of the three planes tangent to $S$ at the vertices of $T(D)$. In this case we shall write $T$ for $T(D)$ and $D(T)$ for $D$, $A(T)$ for $A(D)$, $l(T)$ for $l(D)$.

(iv) *If the area of $T$ is fixed, then the value of $l(T)$ is least when $T$ is an equilateral triangle.*

Write
$$l_A^* = \inf l(T) \quad (A(T) = A).$$

There exists a sequence of triangles $T_i$ such that

$$A(T_i) = A, \qquad l(T_i) \to l_A^* \quad \text{as } i \to \infty.$$

Let the vertices of $T_i$ be $M_1(i)$, $M_2(i)$, $M_3(i)$ and the angle of $T_i$ at the vertex $M_j(i)$ be $\alpha_j(i)$ ($j = 1, 2, 3$).

We may suppose, by the Blaschke selection theorem, that the triangles $T_i$ converge to a set $X$, and that $M_j(i) \to M_j$ as $i \to \infty$. The set $X$ is of area $A$ and is bounded by arcs of at most three great circles. Thus it is either a spherical triangle or a lune (it cannot be a hemisphere since $A$ is the area of a spherical triangle and therefore is less than $2\pi$). If $X$ were a lune, then at least one of the angles $\angle M_1 M_2 M_3$, $\angle M_1 M_3 M_2$, $\angle M_2 M_1 M_3$ would be equal to $\frac{1}{2}A$: say $\angle M_1 M_3 M_2 = \frac{1}{2}A$. Now

$$\tfrac{1}{2}A \leqslant \alpha_j(i), \qquad \alpha_1(i)+\alpha_2(i)+\alpha_3(i) = \pi+A, \qquad \alpha_3(i) \to \tfrac{1}{2}A,$$

$$\cos\{\alpha_1(i)-\tfrac{1}{2}A\}+\cos\{\alpha_2(i)-\tfrac{1}{2}A\}$$
$$= 2\cos\tfrac{1}{2}\{\alpha_1(i)+\alpha_2(i)-A\}\cos\tfrac{1}{2}\{\alpha_1(i)-\alpha_2(i)\} > 0.$$

Thus

$$\lim_{i\to\infty}[\cos\{\alpha_1(i)-\tfrac{1}{2}A\}+\cos\{\alpha_2(i)-\tfrac{1}{2}A\}+\cos\{\alpha_3(i)-\tfrac{1}{2}A\}] \geqslant 1.$$

and $\lim l(T_i) = \infty$. This is not so; thus $X$ is not a lune.

$X$ is a spherical triangle, and we show next that it is an equilateral triangle. $X$ is itself a possible triangle $T$; thus $l(T)$ has a minimum subject to $A(T) = A$. Let the angle of $X$ at $M_i$ be $\alpha_i^*$. Then by the method of Lagrangian undetermined multipliers the function

$$f = \left\{\frac{\sin\frac{1}{2}A}{\sin(\alpha_1-\frac{1}{2}A)\sin(\alpha_2-\frac{1}{2}A)\sin(\alpha_3-\frac{1}{2}A)}\right\}^{\frac{1}{2}} \times$$
$$\times\{\cos(\alpha_1-\tfrac{1}{2}A)+\cos(\alpha_2-\tfrac{1}{2}A)+\cos(\alpha_3-\tfrac{1}{2}A)\}+\lambda(\alpha_1+\alpha_2+\alpha_3-\pi-A)$$

has a minimum at $\alpha_i = \alpha_i^*$. Thus, differentiating with respect to $\alpha_1$, we have

$$\frac{\partial f}{\partial \alpha_1} = \lambda+\frac{(\sin\frac{1}{2}A)^{\frac{1}{2}}}{\{\sin(\alpha_1-\frac{1}{2}A)\sin(\alpha_2-\frac{1}{2}A)\sin(\alpha_3-\frac{1}{2}A)\}^{\frac{1}{2}}} \times$$
$$\times\left[-\sin(\alpha_1-\tfrac{1}{2}A)-\frac{1}{2}\frac{\cos(\alpha_1-\frac{1}{2}A)}{\sin(\alpha_1-\frac{1}{2}A)}\{\cos(\alpha_1-\tfrac{1}{2}A)+\right.$$
$$\left.+\cos(\alpha_2-\tfrac{1}{2}A)+\cos(\alpha_3-\tfrac{1}{2}A)\}\right].$$

There is a similar expression for $\partial f/\partial\alpha_2$ and, when $\alpha_i = \alpha_i^*$ ($i = 1, 2, 3$), we have $\partial f/\partial\alpha_i = 0$ ($i = 1, 2, 3$). Thus, equating two values of $\lambda$, and writing $\beta_i^*$ for $\alpha_i^*-\frac{1}{2}A$ ($i = 1, 2, 3$), we have

$$\sin\beta_1^*+\tfrac{1}{2}\cot\beta_1^*(\cos\beta_1^*+\cos\beta_2^*+\cos\beta_3^*)$$
$$= \sin\beta_2^*+\tfrac{1}{2}\cot\beta_2^*(\cos\beta_1^*+\cos\beta_2^*+\cos\beta_3^*).$$

Hence

$$2\sin\beta_1^*\sin\beta_2^*(\sin\beta_1^*-\sin\beta_2^*) = \sin(\beta_1^*-\beta_2^*)(\cos\beta_1^*+\cos\beta_2^*+\cos\beta_3^*).$$

Therefore

$$2\sin\beta_1^*\sin\beta_2^*\,2\sin\tfrac{1}{2}(\beta_1^*-\beta_2^*)\cos\tfrac{1}{2}(\beta_1^*+\beta_2^*)$$
$$= 2\sin\tfrac{1}{2}(\beta_1^*-\beta_2^*)\cos\tfrac{1}{2}(\beta_1^*-\beta_2^*)(\cos\beta_1^*+\cos\beta_2^*+\cos\beta_3^*),$$

and this implies that either $\sin\frac{1}{2}(\beta_1^*-\beta_2^*) = 0$ or

$$\frac{2\sin\beta_1^*\sin\beta_2^*\cos\frac{1}{2}(\beta_1^*+\beta_2^*)}{\cos\frac{1}{2}(\beta_1^*-\beta_2^*)} = \cos\beta_1^*+\cos\beta_2^*+\cos\beta_3^*$$
$$= 2\cos\tfrac{1}{2}(\beta_1^*+\beta_2^*)\cos\tfrac{1}{2}(\beta_1^*-\beta_2^*)+\cos\beta_3^*.$$

Thus either $\beta_1^* = \beta_2^*$ or

$$\cos\beta_3^* = \frac{2\cos\frac{1}{2}(\beta_1^*+\beta_2^*)}{\cos\frac{1}{2}(\beta_1^*-\beta_2^*)}\{\sin\beta_1^*\sin\beta_2^*-\cos^2\tfrac{1}{2}(\beta_1^*-\beta_2^*)\}$$
$$= \frac{2\cos\frac{1}{2}(\beta_1^*+\beta_2^*)}{\cos\frac{1}{2}(\beta_1^*-\beta_2^*)}[\sin\beta_1^*\sin\beta_2^*-\tfrac{1}{2}\{1+\cos(\beta_1^*-\beta_2^*)\}]$$
$$= -\frac{\cos\frac{1}{2}(\beta_1^*+\beta_2^*)}{\cos\frac{1}{2}(\beta_1^*-\beta_2^*)}\{1+\cos(\beta_1^*+\beta_2^*)\}. \tag{17}$$

Now $\alpha_1^*+\alpha_2^*+\alpha_3^* = \pi+A$ and, since the spherical triangle is contained in a lune whose angle is $\alpha_1^*$, $\alpha_1^* > \frac{1}{2}A$; similarly $\alpha_2^* > \frac{1}{2}A$, $\alpha_3^* > \frac{1}{2}A$. Thus

$$0 < \beta_i^* < \pi \quad (i = 1, 2, 3), \qquad -\tfrac{1}{2}\pi < \tfrac{1}{2}(\beta_1^*-\beta_2^*) < \tfrac{1}{2}\pi,$$
$$\beta_1^*+\beta_2^* = \alpha_1^*+\alpha_2^*-A = \pi-\alpha_3^* < \pi.$$

It follows that the expression for $\cos\beta_3^*$ on the right-hand side of the above equation is negative.

Thus $\beta_3^* > \frac{1}{2}\pi$, and we have shown that, if $\beta_1^* \neq \beta_2^*$, then $\beta_3^* > \frac{1}{2}\pi$.

Now of the three angles $\beta_1^*, \beta_2^*, \beta_3^*$, at most one is greater than $\frac{1}{2}\pi$ since the sum of any two of them is less than $\pi$. Suppose then that $\beta_1^* < \frac{1}{2}\pi$, $\beta_2^* < \frac{1}{2}\pi$. Apply the above argument with $\beta_2^*$ and $\beta_3^*$ instead of $\beta_1^*$ and $\beta_3^*$, and then with $\beta_1^*$ and $\beta_3^*$ instead of $\beta_1^*$ and $\beta_2^*$. We conclude that

$$\beta_2^* = \beta_3^* \quad \text{since } \beta_1^* < \tfrac{1}{2}\pi, \qquad \beta_1^* = \beta_3^* \quad \text{since } \beta_2^* < \tfrac{1}{2}\pi.$$

Thus finally $\beta_1^* = \beta_2^* = \beta_3^*$ and $X$ is equilateral.

This completes the proof of (iv).

(v) *If $T$ is the triangle $M_0 M_2 M_3$ on $S$ and $T'$ is the isosceles triangle $M_0' M_2 M_3$, where $M_0' M_2 = M_0' M_3$ and $A(T) = A(T')$, and if further the base angles of $T'$ are greater than $\frac{1}{2}\pi$, then $l(T) \geqslant l(T')$. (This result is not true without the restriction on the base angles of $T'$.)*

The proof is similar to that of (iv) but is more involved because there are now two conditions on the triangles.

Let $\mathscr{T}$ be the class of triangles $M_1 M_2 M_3$ each with an area equal to that of $T$ and such that $M_1$ lies on the same side of the great circle $M_2 M_3$

as does $M_0$. By Lexell's theorem [see (3) 23], the point $M_1$ lies on the circular arc through $M_0$ that terminates at $M_2'$ and $M_3'$, the points on $S$ diametrically opposite to $M_2$ and $M_3$. As $M_1$ tends to $M_2'$, the triangle $M_1 M_2 M_3$ tends to a lune and the corresponding value of $l(T)$ to infinity as in (iv). Thus there is a point $M_1^*$ such that, if $T^*$ is the triangle $M_1^* M_2 M_3$, then

$$l(T^*) = \inf l(T_1), \qquad T_1 = M_1 M_2 M_3, \qquad A(T_1) = A(T).$$

Our aim is to show that $T^*$ is an isosceles triangle with $M_1^* M_2 = M_1^* M_3$, i.e. that $T^*$ is $T'$ (or the reflection of $T'$ in the great circle $M_2 M_3$).

For the variable triangle $M_1 M_2 M_3$ denote the angles at $M_1$, $M_2$, $M_3$ by $\alpha_1$, $\alpha_2$, $\alpha_3$ respectively and the particular values of these angles when $M_1 = M_1^*$ by $\alpha_1^*$, $\alpha_2^*$, $\alpha_3^*$. Write $A$ for $A(T)$.

Consider next the angle $\alpha_1$. As $M_1$ tends to either $M_2'$ or $M_3'$, $\alpha_1$ tends to $\frac{1}{2}A$ and always $\alpha_1 > \frac{1}{2}A$. Thus there is a position of $M_1$, say $M_1^0$, at which $\alpha_1$ attains its maximum value $\alpha_1^0$. Denote the corresponding values of $\alpha_2$ and $\alpha_3$ by $\alpha_2^0$, $\alpha_3^0$. Then, for appropriate multipliers $\lambda$, $\mu$, the function

$$\alpha_1+\lambda\left\{\frac{\sin(\alpha_1-\frac{1}{2}A)\sin\frac{1}{2}A}{\sin(\alpha_2-\frac{1}{2}A)\sin(\alpha_3-\frac{1}{2}A)}-\tan^2\tfrac{1}{2}\angle M_2 O M_3\right\}+\mu(\alpha_1+\alpha_2+\alpha_3-\pi-A)$$

has a stationary value at $\alpha_1^0$, $\alpha_2^0$, $\alpha_3^0$. Differentiate successively with respect to $\alpha_2$ and $\alpha_3$. Then eliminating $\lambda/\mu$ we obtain

$$\frac{\sin\frac{1}{2}A\sin(\alpha_1^0-\frac{1}{2}A)}{\sin(\alpha_2^0-\frac{1}{2}A)\sin(\alpha_3^0-\frac{1}{2}A)}\{\cot(\alpha_3^0-\tfrac{1}{2}A)-\cot(\alpha_2^0-\tfrac{1}{2}A)\} = 0. \tag{18}$$

Thus $\alpha_3^0 = \alpha_2^0$ and the maximal value of $\alpha_1$ occurs when $M_1 M_2 M_3$ is isosceles. By hypothesis $\alpha_2^0 > \frac{1}{2}\pi$, and thus $\alpha_1^0 < A$. Hence, for any triangle $M_1 M_2 M_3$ of class $\mathscr{T}$, we have $\alpha_1 < A$.

Write $\beta_i$ for $\alpha_i-\frac{1}{2}A$ and $\beta_i^*$ for $\alpha_i^*-\frac{1}{2}A$ ($i = 1, 2, 3$). The function $l(T)$ is a constant multiple of $(\cos\beta_1+\cos\beta_2+\cos\beta_3)/\sin\beta_1$ and we wish to minimize this function with the conditions

$$\frac{\sin\frac{1}{2}A\sin\beta_1}{\sin\beta_2\sin\beta_3} = \tan^2\tfrac{1}{2}\angle M_2 O M_3, \qquad \beta_1+\beta_2+\beta_3 = \pi-\tfrac{1}{2}A.$$

Thus, for appropriate multipliers $\lambda$, $\mu$, the function

$$\frac{\cos\beta_1+\cos\beta_2+\cos\beta_3}{\sin\beta_1}+\lambda(\beta_1+\beta_2+\beta_3-\pi+\tfrac{1}{2}A)+$$
$$+\mu\left(\frac{\sin\frac{1}{2}A\sin\beta_1}{\sin\beta_2\sin\beta_3}-\tan^2\tfrac{1}{2}\angle M_2 O M_3\right)$$

has a stationary value at $\beta_i = \beta_i^*$ ($i = 1, 2, 3$). Differentiating succes-

sively with respect to $\beta_1$, $\beta_2$, $\beta_3$, we obtain three equations for 1, $\lambda$, $\mu$ that are consistent only if

$$\begin{vmatrix} -1-(\cos\beta_1^*+\cos\beta_2^*+\cos\beta_3^*)\dfrac{\cos\beta_1^*}{\sin^2\beta_1^*} & 1 & \dfrac{\cos\beta_1^*}{\sin\beta_2^*\sin\beta_3^*} \\ \dfrac{-\sin\beta_2^*}{\sin\beta_1^*} & 1 & \dfrac{-\sin\beta_1^*\cos\beta_2^*}{\sin^2\beta_2^*\sin\beta_3^*} \\ \dfrac{-\sin\beta_3^*}{\sin\beta_1^*} & 1 & \dfrac{-\sin\beta_1^*\cos\beta_3^*}{\sin\beta_2^*\sin^2\beta_3^*} \end{vmatrix} = 0,$$

i.e.

$$\begin{vmatrix} \dfrac{1+\cos\beta_1^*(\cos\beta_2^*+\cos\beta_3^*)}{\sin\beta_1^*} & 1 & \cot\beta_1^* \\ \sin\beta_2^* & 1 & -\cot\beta_2^* \\ \sin\beta_3^* & 1 & -\cot\beta_3^* \end{vmatrix} = 0.$$

Multiply the last column by $\cos\beta_2^*+\cos\beta_3^*$ and subtract from the first; then multiply the first, second, third rows by $\sin\beta_1^*$, $\sin\beta_2^*$, $\sin\beta_3^*$ respectively. We obtain

$$\begin{vmatrix} 1 & \sin\beta_1^* & \cos\beta_1^* \\ 1+\cos\beta_2^*\cos\beta_3^* & \sin\beta_2^* & -\cos\beta_2^* \\ 1+\cos\beta_2^*\cos\beta_3^* & \sin\beta_3^* & -\cos\beta_3^* \end{vmatrix} = 0,$$

i.e.

$$\sin(\beta_3^*-\beta_2^*)+\sin\beta_1^*(1+\cos\beta_2^*\cos\beta_3^*)(\cos\beta_3^*-\cos\beta_2^*)+$$
$$+\cos\beta_1^*(1+\cos\beta_2^*\cos\beta_3^*)(\sin\beta_3^*-\sin\beta_2^*) = 0,$$

i.e.

$$\sin(\beta_3^*-\beta_2^*)+(1+\cos\beta_2^*\cos\beta_3^*)\{\sin(\beta_1^*+\beta_3^*)-\sin(\beta_1^*+\beta_2^*)\} = 0. \quad (19)$$

Now, since $\alpha_1^* < A$, $\qquad \beta_1^* < \frac{1}{2}A.$

Then $\qquad 2\beta_1^*+\beta_2^*+\beta_3^* < \frac{1}{2}A+\pi-\frac{1}{2}A = \pi.$

Thus $\qquad \beta_1^*+\beta_3^* < \pi-(\beta_1^*+\beta_2^*).$

If now $\beta_3^* > \beta_2^*$, then

$$0 < \beta_2^* < \beta_3^* < \pi-\beta_2^*$$

and so $\sin(\beta_3^*-\beta_2^*) > 0$. Also $\beta_1^*+\beta_2^* < \beta^*+\beta_3^* < \pi-(\beta_1^*+\beta_2^*)$. Hence

$$\sin(\beta_1^*+\beta_3^*)-\sin(\beta_1^*+\beta_2^*) > 0, \qquad 1+\cos\beta_2^*\cos\beta_3^* \geqslant 0.$$

It follows that the expression on the left-hand side of (19) is positive if $\beta_3^* > \beta_2^*$. Similarly it is negative if $\beta_3^* < \beta_2^*$. Thus (19) implies that $\beta_3^* = \beta_2^*$, and this is the required relation.

The value of $l(T)$ when $T$ is an equilateral triangle of area $A$ is equal to

$$3(\sin\tfrac{1}{2}A)^{\frac{1}{2}}\tan\tfrac{1}{6}(A+\pi)\{\sec\tfrac{1}{6}(A+\pi)\}^{\frac{1}{2}}. \quad (20)$$

This function is of fundamental importance in the sequel, and we denote it by $g(A)$. An alternative expression for it is

$$g(A) = 3\tan\tfrac{1}{6}(A+\pi)\{4\sin^2\tfrac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}. \tag{21}$$

(vi) *$g(A)$ is an increasing function of $A$ ($0 \leqslant A \leqslant 2\pi$). There is a certain value of $A$, say $A_1$, such that, for $0 \leqslant A \leqslant A_1$, $g(A)$ is concave and for $A_1 \leqslant A \leqslant 2\pi$, $g(A)$ is convex.*

That $g(A)$ is increasing is trivial. The existence of $A_1$ is established in the Appendix. Numerically $A_1$ lies between $\cdot 54$ and $\cdot 58$.

(vii) *There exists a number $A_2$ ($0 < A_2 < 2\pi$), such that the function $A^{-1}g(A)$ is a decreasing function of $A$ if $0 < A < A_2$ and increasing if $A_2 < A < 2\pi$.*

The proof is given in the Appendix. Calculation shows that the minimum of $A^{-1}g(A)$ is greater than $1\cdot 9$, and that $A_2$ is approximately $1\cdot 42165$.

We next use these properties of $g(A)$ and (iv) to obtain information about the triangles $T\{D(i)\}$ corresponding to the vertices of the extremal polyhedron $P$.

(viii) *The total sum of the functions $l\{D(i)\}$ for those spherical triangles $T\{D(i)\}$ for which $l\{D(i)\} \geqslant 2\cdot 13A\{D(i)\}$ is at most $\cdot 1525$ and the corresponding sum of the $A\{D(i)\}$ is at most $\cdot 015$.*

Let $\theta(K)$ and $\phi(K)$ denote the total area and the sum of the $l\{D(i)\}$ for those triangles $T\{D(i)\}$ for which $l\{D(i)\} \geqslant KA\{D(i)\}$. Since the total length of edges of $P$ is at most 24 and since any triangle $T\{D(j)\}$ satisfies $l\{D(j)\} \geqslant 1\cdot 9A\{D(j)\}$, we have

$$\{4\pi-\theta(K)\}1\cdot 9+\phi(K) \leqslant 24. \tag{22}$$

Since $\phi(K) \geqslant K\,\theta(K)$, we can deduce from this inequality that

$$\theta(K)(K-1\cdot 9) \leqslant 24-4\pi\,1\cdot 9 \leqslant \cdot 124. \tag{23}$$

For $K \geqslant 2\cdot 13$ we have

$$\theta(K) \leqslant \cdot 54 < A_1 < A_2.$$

Now for each of the triangles $T\{D(i)\}$ that compose $\theta(K)$ we have

$$l\{D(i)\} \geqslant g[A\{D(i)\}]$$

and by (vii), since $A\{D(i)\} \leqslant \theta(K) < A_2$,

$$\frac{g[A\{D(i)\}]}{A\{D(i)\}} \geqslant \frac{g\{\theta(K)\}}{\theta(K)}.$$

Thus in (23) we can replace $K$ by $g\{\theta(K)\}/\theta(K)$ to obtain

$$g\{\theta(K)\}-1\cdot 9\theta(K) \leqslant \cdot 124. \tag{24}$$

Now the function $g(A)$ is concave for $A$ satisfying $0 \leqslant A \leqslant A_1$, and a direct evaluation shows that the expression on the left-hand side of (24) is greater than $\cdot 124$ for $\theta(K) = \cdot 54$ and for $\theta(K) = \cdot 015$. Hence (24) can be correct only if $\theta(K) \leqslant \cdot 015$.

Substitution for $\theta(K)$ in (22) leads to $\phi(K) \leqslant \cdot 1525$.

(ix) *Any triangle $T = T\{D(i)\}$ with at least one side of angular length less than or equal to $11^\circ\ 30'$ satisfies $l\{D(i)\} \geqslant 2\cdot 13 A\{D(i)\}$.*

Direct calculation shows that $A^{-1}g(A) > 2\cdot 13$ if $A = \cdot 576$ or if $A = 3$. Thus, by (vii), $A^{-1}g(A) > 2\cdot 13$ if $A \leqslant \cdot 576$ or if $A \geqslant 3$. We shall suppose that $T\{D(i)\}$ has vertices $M_1 M_2 M_3$, and it is sufficient to consider the case when the area of $M_1 M_2 M_3$ lies between $\cdot 576$ and 3. Suppose that $M_1 M_2$ is of angular length $\psi$ and $\psi \leqslant 11^\circ\ 30'$. Let $M_1 M_2 M_3' = T'$ be an isosceles triangle on $M_1 M_2$ as base and with area equal to that of $M_1 M_2 M_3$ and let $M_1 M_2 M_3''$ be an isosceles triangle on $M_1 M_2$ as base with both the base angles equal to $\frac{1}{2}\pi$. The area of $M_1 M_2 M_3''$ is $\psi$ and

$$\psi \leqslant \cdot 576 < \text{area}\, M_1 M_2 M_3'.$$

Thus both the base angles of the triangle $M_1 M_2 M_3'$ are greater than $\frac{1}{2}\pi$. Hence, by (v), $l(T) \geqslant l(T')$ and we need only to consider the case when $T$ coincides with $T'$, i.e. $T$ is isosceles.

We suppose then that $T = M_1 M_2 M_3$ is an isosceles triangle with $M_3 M_1 = M_3 M_2, 3 \geqslant A(T) \geqslant \cdot 576, \angle M_2 O M_1 = \psi = 11^\circ\ 30'$. Then write

$$\alpha_1 = \angle M_1 M_3 M_2, \qquad \alpha_2 = \angle M_3 M_1 M_2 = \angle M_3 M_2 M_1,$$
$$\beta_i = \alpha_i - \tfrac{1}{2}A(T) \quad (i = 1, 2).$$

Then $$\beta_1 + 2\beta_2 = \pi - \tfrac{1}{2}A(T), \qquad \sin\beta_1 = \frac{\tan^2 \frac{1}{2}\psi}{\sin\frac{1}{2}A}\sin^2\beta_2. \tag{25}$$

Thus, from (15),

$$\frac{l(T)}{A(T)} = \frac{\{\sin\frac{1}{2}A(T)\}^{\frac{1}{2}}(\cos\beta_1 + 2\cos\beta_2)}{A(T)(\sin\beta_1)^{\frac{1}{2}}\sin\beta_2} = \frac{\sin\frac{1}{2}A(T)(\cos\beta_1 + 2\cos\beta_2)}{A(T)\tan\frac{1}{2}\psi\sin^2\beta_2}.$$

Since $\alpha_2 > \frac{1}{2}\pi$, we have $\alpha_1 < A(T)$ and $\beta_1 < \frac{1}{2}A(T)$. Thus from

$$\beta_1 + 2\beta_2 = \pi - \tfrac{1}{2}A(T),$$
$$\beta_2 > \tfrac{1}{2}\{\pi - A(T)\}, \qquad \beta_1 + \beta_2 = \pi - \tfrac{1}{2}A(T) - \beta_2 < \tfrac{1}{2}\pi.$$

Hence, since $\beta_1 < \beta_2$ by (25), it follows that $\cos\beta_1 > \sin\beta_2$ and *a fortiori* $\cos\beta_1 + 2\cos\beta_2 > \sin^2\beta_2$. Thus

$$\frac{l(T)}{A(T)} > \frac{1}{\pi\tan\frac{1}{2}\psi} \geqslant 3.$$

This implies the required relation.

(x) *Any triangle $T$ with at least one side of angular length greater than $\chi$ satisfies $l(T) > \tan\frac{1}{2}\chi$.*

Denote the angles of $T$ by $\alpha_1$, $\alpha_2$, $\alpha_3$ and suppose that $\alpha_1$ is opposite a side of length $\chi_1$ ($\chi_1 > \chi$). Then [see (iii) equation (16)]

$$l(T) = \tan\tfrac{1}{2}\chi_1\left\{\frac{\cos(\alpha_1-\frac{1}{2}A)+\cos(\alpha_2-\frac{1}{2}A)+\cos(\alpha_3-\frac{1}{2}A)}{\sin(\alpha_1-\frac{1}{2}A)}\right\},$$

where $A = A(T)$. Thus, since $\frac{1}{2}A < \alpha_2$, $\frac{1}{2}A < \alpha_3$, $\alpha_2+\alpha_3-A < \pi$,

$$\begin{aligned} l(T) &> \tan\tfrac{1}{2}\chi_1\left\{\frac{\cos(\alpha_1-\frac{1}{2}A)+1+\cos(\alpha_2+\alpha_3-A)}{\sin(\alpha_1-\frac{1}{2}A)}\right\} \\ &= \tan\tfrac{1}{2}\chi_1\left\{\frac{1+2\sin\frac{1}{4}A\cos\frac{1}{2}(\alpha_1-\alpha_2-\alpha_3+\frac{1}{2}A)}{\sin(\alpha_1-\frac{1}{2}A)}\right\} \\ &> \frac{\tan\frac{1}{2}\chi_1}{\sin(\alpha_1-\frac{1}{2}A)} > \tan\tfrac{1}{2}\chi_1, \end{aligned} \tag{26}$$

and this is the required result.

COROLLARY. *If $T$ has a side of angular length greater than $11°\ 30'$ or greater than $20°$, then $l(T)$ is greater than $\cdot 09$ or $\cdot 176$ as the case may be.*

(xi) *Every triangle $T\{D(i)\}$ corresponding to a vertex of the extremal polyhedron $P$ satisfies $l\{D(i)\} \leqslant 2{\cdot}13A\{D(i)\}$.*

There must be some triangles which satisfy this inequality since

$$\sum l\{D(i)\} \leqslant 24, \qquad \sum A\{D(i)\} = 4\pi, \qquad 24 \leqslant 2\times 4\pi.$$

If there were some triangles which did not satisfy this inequality, then we could divide the class of triangles $T\{D(i)\}$ into two classes: $C_1$ consisting of those triangles which do not satisfy the inequality and $C_2$ consisting of those which do satisfy the inequality.

By (ix) every side of every triangle $C_2$ has angular length greater than $11°\ 30'$. If there are two or more triangles of $C_1$, then there are at least two which have a side in common with some triangle of $C_2$, i.e. a side of angular length at least $11°\ 30'$. Thus, from the corollary to (x),

$$\sum_{T\{D(i)\}\in C_1} l\{D(i)\} \geqslant 2\times\cdot 09 = \cdot 18.$$

But this is in contradiction with (viii). Thus $C_1$ cannot have more than one member.

Suppose that $C_1$ has one member, say $T$. Every side of $T$ has angular length greater than $11°\ 30'$. We shall show that this implies that either $A(T) > \cdot 015$ or the largest side of $T$ has angular length greater than $20°$. Let the vertices of $T$ be $M_1$, $M_2$, $M_3$ and let $M_1 M_2$ be the largest side of $T$.

We assume that the angular length of $M_1 M_2$ is less than 20°. We consider the class of all the triangles $MM_1 M_2$, where $M$ is any point such that the angular distances $MM_1$ and $MM_2$ are not less than $11° 30'$ and not more than $M_1 M_2$. There is a member of this class which has the least area. Denote it by $M_1 M_2 M_3^*$. Then $M_1 M_2 M_3^*$ is a genuine triangle and $M_1 M_3^* = M_3^* M_2 = 11° 30'$. For, if (say) $M_1 M_3^* > 11° 30'$, we could give $M_3^*$ slight variations such that $M_2 M_3^*$ remains constant, i.e. we could move $M_3^*$ on the small circle $\gamma$ with centre $M_2$ and radius $M_2 M_3^*$. Now the circle $M_3^* M_1' M_2'$, where $M_1'$, $M_2'$ are the reflections of $M_1$ and $M_2$ in $O$, is either tangential and exterior to $\gamma$ or cuts $\gamma$. In either case we can move $M_3^*$ along $\gamma$ in such a direction that the area of $M_1 M_2 M_3^*$ is reduced. This is impossible; thus $M_1 M_3^* = 11° 30'$ and similarly $M_2 M_3^* = 11° 30'$. Let $N$ be the mid-point of $M_1 M_2$. Then, if we write $c$ for the angular length $M_1 M_3^*$ and $\alpha$ for angle $M_3^* M_1 M_2$, we have by the well-known formulae of spherical trigonometry

$$\text{area}\, M_1 M_2 M_3^* = 2\,\text{area}\, M_1 N M_3^* = 2\{\alpha - \tan^{-1}(\cos c \tan \alpha)\}.$$

As $\alpha$ increases, with $c$ kept constant, this last expression increases to a maximum at $\cos c \tan^2\alpha = 1$ and then decreases. Thus the least possible value of the area $M_1 M_2 M_3^*$ occurs when either $M_1 M_2 = 20°$ or $M_1 M_2 = 11° 30'$. Calculation shows that in both of these cases

$$\text{area}\, M_1 M_2 M_3^* > \cdot 015.$$

But now we have a contradiction. If $T$, the one member of $C_1$, has its largest side of length less than 20°, then by the above the area of $T$ exceeds $\cdot 015$, in contradiction with (viii). On the other hand, if the largest side-length of $T$ is not less than 20°, then by the corollary to (x), $l(T) > \cdot 155$ which is again in contradiction with (viii). Thus in fact $C_1$ cannot have exactly one member and must be empty. Thus (xi) is proved.

We can now complete the argument and show that $P$ is a cube. By (xi) every triangle $T\{D(i)\}$ satisfies $l\{D(i)\} \leqslant 2{\cdot}13 A\{D(i)\}$ and hence by evaluating $g(A)$ at $A = A_1$ it follows that $A\{D(i)\} > A_1$ for every vertex $D(i)$ of $P$. But $g(A)$ is convex for $A$ in the range $A_1 \leqslant A \leqslant 2\pi$ and hence, if $P$ has $k$ vertices,

$$24 \geqslant \sum l\{D(i)\} \geqslant \sum g[A\{D(i)\}] \geqslant kg\left[\frac{\sum A\{D(i)\}}{k}\right] = kg\left(\frac{4\pi}{k}\right). \quad (27)$$

Calculation shows that this inequality is possible only if $k = 8$ or $k = 9$. Since three edges of $P$ meet at each vertex and the number of edge-vertex incidences is even, it follows that the number of vertices of $P$ is even. Thus $P$ cannot have 9 vertices and it accordingly has 8 vertices. But

then $8g(\frac{1}{2}\pi) = 24$, and thus equality holds throughout (27). Hence each $T\{D(i)\}$ is an equilateral triangle of area $\frac{1}{2}\pi$, and $P$ is a cube.

Finally we have to show that a cube is the only type of extremal polyhedron. An examination of the argument shows that, if there is any other type of extremal polyhedron, it arises only because of the assumption made in (i): that every face-plane of $P$ touches $S$. Therefore any other extremal polyhedron can be obtained from $P$ by translating some of the face-planes of $P$ so that they no longer touch $S$. But these processes obviously do not lead to another extremal polyhedron since when applied to a cube they lead to a polyhedron with a larger value of $L$. Thus the cube is the only polyhedron which circumscribes a sphere of unit radius and has total edge-length not exceeding 24.

## Appendix

*Proof of* (vi). We use the second form for $g(A)$. Then

$$g(A) = 3\tan\{\tfrac{1}{6}(A+\pi)\}\{4\sin^2\tfrac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}.$$

Writing $x = \sin^2\frac{1}{6}(A+\pi)$ and $y = 4x-1$, we see that

$$\frac{d}{dx}\left(\frac{dg}{dA}\right) = \frac{y^3-3y^2+27y-9}{y^{\frac{3}{2}}(y-3)^2}. \tag{28}$$

Since $0 \leqslant A \leqslant 2\pi$, $\frac{1}{6}(A+\pi)$ lies between $\frac{1}{6}\pi$ and $\frac{1}{2}\pi$. Thus $\frac{1}{4} \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 3$. The expression on the right-hand side of (28) has a unique zero, at say $y_0$, and is positive if $y > y_0$ and negative if $y < y_0$. Rewriting the expression $y^3-3y^2+27y-9$ as $(y-1)^3+24y-8$ we see that

$$\tfrac{1}{3} < y_0 < \tfrac{1}{3}+\tfrac{1}{81}, \qquad \cdot 54 < A_1 < \cdot 58.$$

*Proof of* (vii). We again use the second form for $g(A)$ so that

$$\frac{g(A)}{A} = \frac{3\tan\frac{1}{6}(A+\pi)}{A}\{4\sin^2\tfrac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}.$$

Then

$$\frac{d}{dA}\left\{\frac{g(A)}{A}\right\}$$

$$= \left\{\frac{\sec^2\frac{1}{6}(A+\pi)}{2A} - \frac{3\tan\frac{1}{6}(A+\pi)}{A^2}\right\}\{4\sin^2\tfrac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}+$$

$$+\frac{3\tan\frac{1}{6}(A+\pi)\{\frac{2}{3}\sin\frac{1}{6}(A+\pi)\cos\frac{1}{6}(A+\pi)\}}{A\{4\sin^2\frac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}}$$

$$= \frac{1}{2A^2\{4\sin^2\frac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}}[\{4\sin^2\tfrac{1}{6}(A+\pi)-1\}\times$$

$$\times\{A\sec^2\tfrac{1}{6}(A+\pi)-6\tan\tfrac{1}{6}(A+\pi)\}+4A\sin^2\tfrac{1}{6}(A+\pi)]$$

$$= \frac{\sin^2 \frac{1}{6}(A+\pi)}{2A^2\{4\sin^2 \frac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}}\Big[\{4\sin^2 \tfrac{1}{6}(A+\pi)-1\}\times$$

$$\times\left(\frac{A}{\sin^2 \frac{1}{6}(A+\pi)\cos^2 \frac{1}{6}(A+\pi)}-\frac{6}{\sin \frac{1}{6}(A+\pi)\cos \frac{1}{6}(A+\pi)}\right)+4A\Big]$$

$$= \frac{12\sin^2 \frac{1}{6}(A+\pi)}{2A^2\sin^2 \frac{1}{3}(A+\pi)\{4\sin^2 \frac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}}[\{1-2\cos \tfrac{1}{3}(A+\pi)\}\times$$

$$\times\{\tfrac{1}{3}A-\sin \tfrac{1}{3}(A+\pi)\}+\tfrac{1}{3}A\sin^2 \tfrac{1}{3}(A+\pi)]$$

$$= \frac{6\sin^2 \frac{1}{6}(A+\pi)\{1-2\cos \frac{1}{3}(A+\pi)\}}{A^2\sin \frac{1}{3}(A+\pi)\{4\sin^2 \frac{1}{6}(A+\pi)-1\}^{\frac{1}{2}}}\times$$

$$\times\left\{\frac{\frac{1}{3}A}{\sin \frac{1}{3}(A+\pi)}+\frac{\frac{1}{3}A\sin \frac{1}{3}(A+\pi)}{1-2\cos \frac{1}{3}(A+\pi)}-1\right\}.$$

Write

$$h(A) = \frac{\frac{1}{3}A}{\sin \frac{1}{3}(A+\pi)}, \qquad f(A) = \frac{\frac{1}{3}A\sin \frac{1}{3}(A+\pi)}{1-2\cos \frac{1}{3}(A+\pi)}.$$

Then, for $0 < A < 2\pi$, $\dfrac{d}{dA}\left\{\dfrac{g(A)}{A}\right\}$ is a positive multiple of $h(A)+f(A)-1$. Moreover we have

$$\lim_{A\to 0} h(A) = 0, \qquad \lim_{A\to 0} f(A) = \lim \frac{\frac{1}{3}A\sin \frac{1}{3}(A+\pi)}{1-\cos \frac{1}{3}A+\sqrt{3}\sin \frac{1}{3}A} = \frac{1}{2}.$$

Hence $\dfrac{d}{dA}\left\{\dfrac{g(A)}{A}\right\}$ is negative when $A$ is small and positive. Also

$$\lim_{A\to 2\pi} h(A) = +\infty, \qquad \lim_{A\to 2\pi} f(A) = 0.$$

Hence $\dfrac{d}{dA}\left\{\dfrac{g(A)}{A}\right\}$ is positive for values of $A$ near to $2\pi$.

Thus there are two possibilities. Either (i) $\dfrac{d}{dA}\left\{\dfrac{g(A)}{A}\right\}$ has exactly one zero $A_2$ in $0 < A < 2\pi$ and is negative for $A < A_2$ and positive for $A > A_2$ or (ii) $\dfrac{d}{dA}\left\{\dfrac{g(A)}{A}\right\}$ has three or more roots in $0 < A < 2\pi$. (Multiple roots are counted according to their multiplicity.)

Case (i) is exactly the situation that we wish to establish. We shall show that case (ii) cannot occur.

In case (ii) the equation $h(A)+f(A) = 1$,

i.e. $$\tfrac{1}{3}A = \left\{\frac{1}{\sin \frac{1}{3}(A+\pi)}+\frac{\sin \frac{1}{3}(A+\pi)}{1-2\cos \frac{1}{3}(A+\pi)}\right\}^{-1},$$

has three or more roots in $0 < A < 2\pi$, and this means that the second derivative of the function on the right-hand side of this equation has a zero in $0 < A < 2\pi$. We shall show that this is not so.

Write $\phi$ for $\frac{1}{3}(A+\pi)$; then $\frac{1}{3}\pi < \phi < \pi$ and the function concerned is

$$F(\phi) = \frac{(1-2\cos\phi)\sin\phi}{2-2\cos\phi-\cos^2\phi}.$$

Differentiate and write $x$ for $\cos\phi$. Then

$$F'(\phi) = \frac{2-6x^2+5x^3}{(2-2x-x^2)^2},$$

$$F''(\phi) = \frac{d}{dx}\{F'(\phi)\}\frac{dx}{d\phi} = -\frac{\sin\phi\{8-16x+30x^2-22x^3+5x^4\}}{(2-2x-x^2)^3}.$$

Since $\frac{1}{3}\pi < \phi < \pi$, $\sin\phi > 0$ and $\frac{1}{2} > x > -1$. Thus $F''(\phi)$ is negative and non-zero in $0 < A < 2\pi$. Hence case (ii) above cannot occur, and the existence of $A_2$ as required has been established.

Direct substitution shows that $h(A)+f(A)-1$ is negative when $A = 81^\circ\ 12'$ and positive when $A = 81^\circ\ 18'$. Thus $A_2$ lies between these limits and

$$\min\frac{g(A)}{A} = \frac{3\tan\frac{1}{6}(A_2+\pi)(\sin\frac{1}{2}A_2)^{\frac{1}{2}}}{A_2\{\cos\frac{1}{6}(A_2+\pi)\}^{\frac{1}{2}}} > \frac{3\tan 43^\circ\ 32'}{1{\cdot}419}\left(\frac{\sin 40^\circ\ 36'}{\cos 43^\circ\ 32'}\right)^{\frac{1}{2}} > 1{\cdot}9.$$

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# INEQUALITIES FOR GRAM DETERMINANTS

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**1.** THIS note is concerned with some inequalities for Gram determinants of $L^2$ functions. The author is greatly indebted to the referee for considerable improvement in the presentation of the results.

**2.** The symbols $E$, $E'$, $E_1$, $E_2$ will denote measurable linear sets, and it will be assumed that all functions considered below are defined and $L^2$-integrable on the sets in question.†

I shall denote by $\{\phi\}$ the system of complex-valued functions

$$\{\phi_1(x),..., \phi_n(x)\}.$$

This system is said to be *linearly dependent over the set* $E$ if some non-trivial linear combination of $\phi_1,..., \phi_n$ vanishes p.p. on $E$; in the contrary case $\{\phi\}$ is *linearly independent over* $E$. Furthermore, $\phi_n$ is said to be *linearly dependent* on $\phi_1,..., \phi_{n-1}$ over $E$ if $\phi_n$ is equal p.p. on $E$ to some linear combination of $\phi_1,..., \phi_{n-1}$. We note that, if $E \subseteq E'$ and $\{\phi\}$ is linearly independent over $E$, then $\{\phi\}$ is linearly independent over $E'$.

The *Gram matrix*

$$\Gamma(\phi; E) \equiv \Gamma(\phi_1,..., \phi_n; E)$$

of $\{\phi\}$ on the set $E$ is defined as the $n \times n$ matrix whose $(r, s)$th element is

$$\int\limits_E \phi_r \bar{\phi}_s \, dx.$$

The *Gram determinant*

$$\Delta(\phi; E) \equiv \Delta(\phi_1,..., \phi_n; E)$$

of $\{\phi\}$ on $E$ is defined as det $\Gamma(\phi; E)$. Clearly $\Gamma(\phi; E)$ is the matrix of the non-negative hermitian form

$$\int\limits_E |u_1 \phi_1 + ... + u_n \phi_n|^2 \, dx$$

in the variables $u_1,..., u_n$. Hence we have

$$\Delta(\phi; E) \geqslant 0, \tag{2.1}$$

† By an obvious change of notation we can extend all our results to functions of several variables.

and the following statements are plainly equivalent:

(i) the inequality (2.1) is strict;

(ii) $\Gamma(\phi; E)$ is positive definite;

(iii) $\{\phi\}$ is linearly independent over $E$.

Our aim is to establish two inequalities for Gram determinants and to investigate the conditions under which these inequalities are strict.

THEOREM 1. *Let $E_1 \subseteq E_2$ and suppose that $\{\phi\}$ is linearly independent over $E_1$. Then*

$$\Delta(\phi; E_1) \leqslant \Delta(\phi; E_2), \tag{2.2}$$

*and this inequality is strict if and only if*

$$\textit{at least one } \phi_r \textit{ does not vanish p.p. on } E_2 - E_1. \tag{2.3}$$

The relation (2.2) was deduced by Courant and Hilbert [(1) 107] from the mini-max principle for characteristic roots of hermitian matrices. Here a different derivation will be given.

THEOREM 2. *Let $E_1 \subseteq E_2$ and suppose that $\{\phi_1,..., \phi_n\}$ is linearly independent over $E_1$. Then*†

$$\frac{\Delta(\phi_1,..., \phi_{n-1}; E_1)}{\Delta(\phi_1,..., \phi_n; E_1)} \geqslant \frac{\Delta(\phi_1,..., \phi_{n-1}; E_2)}{\Delta(\phi_1,..., \phi_n; E_2)}. \tag{2.4}$$

*If $\phi_n$ is not linearly dependent on $\phi_1,..., \phi_{n-1}$ over $E_2 - E_1$, then the inequality (2.4) is strict.*

We may note that this sufficient condition for strictness is not necessary. This is demonstrated by the example

$$\phi_1(x) = 1 \quad (0 \leqslant x \leqslant 2), \qquad \phi_2(x) = \begin{cases} x & (0 \leqslant x \leqslant 1), \\ 1 & (1 \leqslant x \leqslant 2), \end{cases}$$

$$E_1 = [0, 1], \qquad E_2 = [0, 2].$$

COROLLARY. *Under the conditions of Theorem 2 we have*

$$\frac{\Delta(\phi_1,..., \phi_r; E_1)}{\Delta(\phi_1,..., \phi_n; E_1)} \geqslant \frac{\Delta(\phi_1,..., \phi_r; E_2)}{\Delta(\phi_1,..., \phi_n; E_2)} \quad (r = 1,..., n-1).$$

This follows at once if we write

$$\frac{\Delta(\phi_1,..., \phi_r; E_k)}{\Delta(\phi_1,..., \phi_n; E_k)} = \prod_{s=r}^{n-1} \frac{\Delta(\phi_1,..., \phi_s; E_k)}{\Delta(\phi_1,..., \phi_{s+1}; E_k)} \quad (k = 1, 2)$$

and apply (2.4) to each factor of the product.

† The denominators in (2.4) are positive in view of the hypothesis.

3. I shall use the symbol $\phi$ to represent the column vector†

$$(\phi_1(x), \phi_2(x), \dots, \phi_n(x))^T.$$

If $P$ is an $n \times n$ matrix, then $\psi = P\phi$ is automatically defined by this convention.

LEMMA 1. *Let $P$ be an $n \times n$ matrix with constant elements. Then*

$$\Gamma(P\phi; E) = P\,\Gamma(\phi; E)\,P^*, \tag{3.1}$$

$$\Delta(P\phi; E) = |\det P|^2 \Delta(\phi; E). \tag{3.2}$$

Writing $\psi = P\phi$ we have

$$\psi\psi^* = P\phi\phi^* P^*.$$

Integrating every element over $E$ we obtain

$$\Gamma(\psi; E) = P\,\Gamma(\phi; E)\,P^*,$$

and the asserted relations follow.

LEMMA 2. *Let $E_1 \subseteq E_2$ and suppose that $\{\phi\}$ is linearly independent over $E_1$. Then there exists a non-singular matrix $P$ with constant elements such that the system $\{\psi\}$, defined by $\psi = P\phi$, is orthonormal on $E_1$ and orthogonal on $E_2$.*

By hypothesis and the equivalence of statements (ii) and (iii) in § 2, it follows that both $\Gamma(\phi; E_1)$ and $\Gamma(\phi; E_2)$ are positive definite hermitian matrices. Hence there exists a non-singular matrix $P$ (with constant elements) such that $P\,\Gamma(\phi; E_1)\,P^*$ is the unit matrix while

$$P\,\Gamma(\phi; E_2)\,P^*$$

is some diagonal matrix, say $\operatorname{diag}(\mu_1, \dots, \mu_n)$. Using (3.1) we therefore have, for $r, s = 1, \dots, n$,

$$\int_{E_1} \psi_r \bar{\psi}_s\,dx = \delta_{rs}, \qquad \int_{E_2} \psi_r \bar{\psi}_s\,dx = \mu_r \delta_{rs}.$$

4. We next come to the proof of Theorem 1. Suppose, in the first place, that (2.3) is not satisfied, i.e. $\phi_1, \dots, \phi_n$ all vanish p.p. on $E_2 - E_1$. Then

$$\int_{E_2 - E_1} \phi_r \bar{\phi}_s\,dx = 0 \quad (r, s = 1, \dots, n)$$

and so $\Gamma(\phi; E_1) = \Gamma(\phi; E_2)$. This implies that there is equality in (2.2).

Next, let (2.3) be satisfied; and let $P$ and $\psi$ have the same meaning as in Lemma 2. Then, by (3.2), we have, for $k = 1, 2$,

$$\begin{aligned}\Delta(\phi; E_k) &= |\det P|^{-2}\,\Delta(\psi; E_k) \\ &= |\det P|^{-2} \prod_{r=1}^{n} \int_{E_k} |\psi_r|^2\,dx.\end{aligned}$$

† Transposes are indicated by a $T$ and transposed conjugates by an asterisk.

Furthermore
$$1 = \int\limits_{E_1} |\psi_r|^2\,dx \leqslant \int\limits_{E_2} |\psi_r|^2\,dx. \tag{4.1}$$

The inequality in (4.1) is strict for at least one value of $r$; otherwise $\psi_1,\ldots,\psi_n$ would all vanish p.p. on $E_2-E_1$ and this is incompatible with (2.3) and the relation $\phi = P^{-1}\psi$. We infer, therefore, that there is strict inequality in (2.2).

**5.** The next three sections contain the proof of Theorem 2. Since $\{\phi_1,\ldots,\phi_{n-1}\}$ is linearly independent over $E_1$, there exists, by Lemma 2, a non-singular $(n-1)\times(n-1)$ matrix $Q$ such that the system $\{\psi_1,\ldots,\psi_{n-1}\}$ defined by the equation
$$(\psi_1,\ldots,\psi_{n-1})^T = Q(\phi_1,\ldots,\phi_{n-1})^T \tag{5.1}$$
is orthogonal over $E_1$ and $E_2$. We put
$$\psi_n = \phi_n + \sum_{s=1}^{n-1} \alpha_s \phi_s \tag{5.2}$$
and seek to determine the constants $\alpha_1,\ldots,\alpha_{n-1}$ such that the augmented system $\{\psi_1,\ldots,\psi_{n-1},\psi_n\}$ is still orthogonal on $E_1$, i.e.
$$\sum_{s=1}^{n-1} \bar{\alpha}_s \int\limits_{E_1} \psi_r \bar{\phi}_s\,dx = -\int\limits_{E_1} \psi_r \bar{\phi}_n\,dx \quad (r = 1,\ldots,n-1). \tag{5.3}$$

But, using (5.1), we have
$$\begin{bmatrix} \psi_1\bar{\phi}_1 & \cdot & \cdot & \cdot & \psi_1\bar{\phi}_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_{n-1}\bar{\phi}_1 & \cdot & \cdot & \cdot & \psi_{n-1}\bar{\phi}_{n-1} \end{bmatrix} = Q \begin{bmatrix} \phi_1\bar{\phi}_1 & \cdot & \cdot & \cdot & \phi_1\bar{\phi}_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{n-1}\bar{\phi}_1 & \cdot & \cdot & \cdot & \phi_{n-1}\bar{\phi}_{n-1} \end{bmatrix}.$$

Therefore, by the equivalence of (i) and (iii) in § 2,
$$\det_{1\leqslant r,s\leqslant n-1}\left(\int\limits_{E_1} \psi_r\bar{\phi}_s\,dx\right) = \det Q\,\Delta(\phi_1,\ldots,\phi_{n-1}; E_1) \neq 0.$$

Thus the equations (5.3) serve to determine uniquely the constants $\alpha_1,\ldots,\alpha_{n-1}$.

**6.** Some further notations will now be required. We first observe that
$$(\psi_1,\ldots,\psi_n)^T = P(\phi_1,\ldots,\phi_n)^T, \tag{6.1}$$
where
$$P = \begin{bmatrix} Q & 0 \\ \alpha & 1 \end{bmatrix}, \qquad \alpha = (\alpha_1,\ldots,\alpha_{n-1}).$$

Put
$$m^{-1} = \det P = \det Q.$$

We write

$$c_r = \int_{E_2} \psi_n \bar{\psi}_r \, dx, \qquad \gamma_r = \int_{E_2 - E_1} \psi_n \bar{\psi}_r \, dx \quad (r = 1,..., n-1),$$

$$d_r = \int_{E_2} |\psi_r|^2 \, dx, \qquad \delta_r = \int_{E_2 - E_1} |\psi_r|^2 \, dx \quad (r = 1,..., n).$$

Since $\{\phi\}$ is linearly independent over $E_1$, it follows that all $d_r$ and all integrals $\int_{E_1} |\psi_r|^2 \, dx$ are positive. Furthermore, since $\{\psi_1,..., \psi_n\}$ is orthogonal on $E_1$,

$$c_r = \gamma_r \quad (r = 1,..., n-1). \tag{6.2}$$

Let $J$ be the (possibly empty) subset of $\{1, 2,..., n-1\}$ such that $\delta_r > 0$ when $r \in J$ and $\delta_r = 0$ otherwise. Let $J'$ be the set consisting of all integers in $J$ together with $n$. We note that

$$\gamma_r = 0 \quad (r \notin J). \tag{6.3}$$

The left-hand side and the right-hand side of (2.4) will be denoted by $\rho_1$, $\rho_2$ respectively.

7. Using (3.2), (5.1), (6.1), and the orthogonality properties of the $\psi$'s, we obtain

$$\rho_1 = \left\{ \int_{E_1} |\psi_n|^2 \, dx \right\}^{-1}, \tag{7.1}$$

$$\Delta(\phi_1,..., \phi_{n-1}; E_2) = m^2 d_1 ... d_{n-1}, \tag{7.2}$$

$$\Delta(\phi_1,..., \phi_n; E_2) = m^2 \begin{vmatrix} d_1 & 0 & . & . & . & 0 & \bar{c}_1 \\ 0 & d_2 & . & . & . & 0 & \bar{c}_2 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & d_{n-1} & \bar{c}_{n-1} \\ c_1 & c_2 & . & . & . & c_{n-1} & d_n \end{vmatrix}. \tag{7.3}$$

By (7.3), (6.2), and (6.3),

$$\Delta(\phi_1,..., \phi_n; E_2) = m^2 d_1 ... d_{n-1} \Big( d_n - \sum_{r=1}^{n-1} d_r^{-1} |\gamma_r|^2 \Big)$$
$$= m^2 d_1 ... d_{n-1} \Big( d_n - \sum_{r \in J} d_r^{-1} |\gamma_r|^2 \Big).$$

Hence, by (7.2),

$$\rho_2 = \left\{ \int_{E_1} |\psi_n|^2 \, dx + \theta \right\}^{-1}, \tag{7.4}$$

where

$$\theta = \delta_n - \sum_{r \in J} d_r^{-1} |\gamma_r|^2.$$

Suppose now that

$$\delta_n > 0. \tag{7.5}$$

If $\gamma_r = 0$ whenever $r \in J$, then $\theta > 0$. If, on the other hand, $\gamma_r \neq 0$ for some $r$ in $J$, then, since $d_r > \delta_r$ $(r = 1,..., n-1)$,

$$\theta > \delta_n - \sum_{r \in J} \delta_r^{-1} |\gamma_r|^2 = \prod_{r \in J} \delta_r^{-1} \Delta(\psi_r, r \in J'; E_2 - E_1),$$

and we again have $\theta > 0$ in view of (2.1). Hence, by (7.1) and (7.4), there is strict inequality in (2.4) if (7.5) is satisfied.

Suppose next that $\delta_n = 0$. Then evidently

$$\gamma_1 = \gamma_2 = \dots = \gamma_{n-1} = 0 \quad \text{and so } \theta = 0.$$

There is therefore equality in (2.4).

We have thus established (2.4) and have shown that the inequality is strict if and only if (7.5) is satisfied.† Now, if $\phi_n$ is not linearly dependent on $\phi_1, \dots, \phi_{n-1}$ over $E_2 - E_1$, then it follows by (5.2) that (7.5) is satisfied. This completes the proof.

[*Added March* 1957]. Since forwarding the manuscript to the Journal, the author has observed that the theorems of this note hold not only for Gram determinants, but for the determinants of any positive definite hermitian matrices. For example, Theorem 1 of this note is equivalent to the corollary to Theorem 13.5.4 of *An Introduction to Linear Algebra* by L. Mirsky (Oxford, 1955). In this aspect Theorem 2 seems to have escaped previous notice.

It is straightforward to prove that every positive definite (or semi-definite) hermitian matrix is the Gram matrix of some system $\{\phi\}$ over any *given* set $E$ (of positive measure). Thus the restatement of the theorems of this note in terms of positive definite hermitian matrices would follow at once.

† There does not seem to be any simple way of expressing this condition in terms of the original system $\{\phi\}$.

## REFERENCE

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# ON HAUSDORFF AND QUASI-HAUSDORFF METHODS OF SUMMABILITY

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## 1. Introduction

THE present paper is a continuation of the earlier papers by me (**16, 17**) on the summability of sequences or series by Hausdorff and quasi-Hausdorff methods. The necessary and sufficient conditions for a quasi-Hausdorff matrix to be regular for series-to-series transformations have been proved by me in (**16**) and the inclusion theorems for these methods have been proved in (**17**). The very close relationship between the Hausdorff and quasi-Hausdorff methods, pointed out in (**16**) and recently, again, by Kuttner (**9**) has suggested the investigation of the problem of comparing the strengths of the two methods for a class of sequences, with suitable restrictions if necessary, and this paper deals with the same and also a few other problems associated therewith.

In § 2 of the paper I prove theorems relating to the regularity and absolute regularity of Hausdorff and quasi-Hausdorff matrices for series-to-series or sequence-to-sequence transformations and also the equivalence of the two methods for the class of bounded sequences which are Borel summable; § 3 is on the strong regularity [defined by Lorentz (**10**)] of these methods and § 4 is on the method $(S^*, \mu)$, defined in the paper, and its relationship to the Hausdorff and quasi-Hausdorff methods.

*Definitions and lemmas*

Let $A \equiv (a_{nk})$ $(n, k = 0, 1, 2,...)$ be a matrix which defines a sequence $\{t_n\}$ in terms of another sequence $\{s_n\}$ by

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k. \tag{1}$$

Then the matrix $A$ provides a *sequence-to-sequence transformation* or a *series-to-sequence transformation* or a *series-to-series* transformation according as it converts $\{s_n\}$ to $\{t_n\}$, or $\sum s_n$ to $\{t_n\}$, or $\sum s_n$ to $\sum t_n$.

The following definitions in general use for the matrix $A$ of sequence-to-sequence transformations can be made applicable, with obvious changes, to other kinds of transformations.

The matrix $A$ is said to be *efficient* for summing $\{s_n\}$ to the sum $l$ if

$\lim t_n = l$ (finite), and the usual symbolical statement of this condition is $A$-$\lim s_n = l$. $A$ is said to be *convergence-preserving* or *conservative* if it transforms every convergent sequence $\{s_n\}$ to a convergent sequence $\{t_n\}$, with limit not necessarily same as that of $\{s_n\}$, while $A$ is said to be *regular* if it transforms every convergent $\{s_n\}$ with $\lim s_n = l$ into a convergent $\{t_n\}$ with the same limit $l$. $A$ is said to be *absolutely conservative* or *preserving absolute convergence* if $\sum |s_n - s_{n-1}| < \infty$ always implies $\sum |t_n - t_{n-1}| < \infty$ and the special class of $A$ for which we have in addition that $\lim s_n = \lim t_n$ is said to be *absolutely regular*.† The various types of matrices occurring in this paper, together with their specifications, stated within brackets, are: (i) $K$-matrix (sequence-to-sequence conservative), (ii) $T$-matrix (sequence-to-sequence regular), (iii) $\beta$-matrix (series-to-sequence conservative), (iv) $\gamma$-matrix (series-to-sequence regular), (v) $\delta$-matrix (series-to-series conservative), and (vi) $\alpha$-matrix (series-to-series regular). A special $\delta$-matrix and a special $\alpha$-matrix, in each of which $\lim a_{nk} = 0$ ($k_n \to \infty$; $n = 0, 1,...$) are called respectively a '$\delta_0$-matrix' and an '$\alpha_0$-matrix'.

The various matrices defined above are characterized by the following lemmas:

Lemma 1. *The matrix $A \equiv (a_{nk})$ is a $K$-matrix if and only if its elements satisfy the conditions*:

(i) $\sup_n \sum_k |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk}$ $(= \delta_k)$ *exists for each fixed* $k$;

(iii) $\gamma_n = \sum_k a_{nk} \to \gamma$ *as* $n \to \infty$.

*The matrix $A$ is a $T$-matrix if and only if, in addition, $\delta_k \equiv 0$ and $\gamma = 1$.*

Lemma 2. *The matrix $G \equiv (g_{nk})$ is a $\beta$-matrix if and only if its elements satisfy the conditions*:

(i) $\sup_n \sum_k |g_{nk} - g_{n,k+1}| < \infty$;

(ii) $\lim_{n\to\infty} g_{nk}$ $(= \beta_k)$ *exists for each fixed* $k$.

*The matrix $G$ is a $\gamma$-matrix if and only if, in addition, $\beta_k \equiv 1$.*

The proof of Lemma 2 has been given by Cooke [(**2**) 65–67].

† It may, however, be noted that an absolutely regular matrix need not be regular, nor need an absolutely conservative matrix be conservative.

LEMMA 3. *The matrix $H \equiv (h_{nk})$ is a δ-matrix or an α-matrix if and only if the matrix $G \equiv (g_{nk})$ defined by*

$$g_{nk} = h_{0k}+h_{1k}+...+h_{nk} \tag{2}$$

*is a β-matrix or a γ-matrix, respectively.*

Lemma 3 is due to Vermes (**18**, **19**).

LEMMA 4. *The series-to-series transformation defined by the matrix $H \equiv (h_{nk})$ is absolutely conservative if and only if*

$$\sup_k \sum_n |h_{nk}| < \infty \tag{3}$$

*and it is absolutely regular if and only if in addition*

$$\sum_n h_{nk} \equiv 1. \tag{4}$$

LEMMA 5. *The sequence-to-sequence transformation defined by the matrix $A \equiv (a_{nk})$ is absolutely conservative if and only if the row sums $\sum_k a_{nk}$ converge for every $n$ and*

$$\sum_{n=1}^{\infty} |g_{nk}-g_{n-1,k}| \leqslant M \quad (k = 0, 1,...), \tag{5}$$

*where $g_{nk} = \sum_{\nu=k}^{\infty} a_{n\nu}$ and $M$ is independent of $k$. It is absolutely regular if and only if in addition*

$$\lim_{n\to\infty} \sum_k a_{nk} = 1 \quad \textit{and} \quad \lim_{n\to\infty} a_{nk} = 0 \quad (k = 0, 1,...).$$

Lemmas 4 and 5, of which the latter is essentially due to Mears (**13**), have been proved by Knopp and Lorentz (**8**).

*Hausdorff and quasi-Hausdorff methods.* The matrix $\lambda \equiv (H, \mu_n)$ defined by

$$\lambda_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k \quad (n \geqslant k), \qquad \lambda_{nk} = 0 \quad (n < k) \tag{6}$$

is said to be a *Hausdorff matrix* and its transpose $\lambda^* \equiv (H^*, \mu_n)$ defined by

$$\lambda^*_{nk} = 0 \quad (n > k), \qquad \lambda^*_{nk} = \binom{k}{n}\Delta^{k-n}\mu_n \quad (n \leqslant k) \tag{7}$$

is said to be a *quasi-Hausdorff matrix*. The well-known Cesàro, Hölder, and Euler transformations are defined by Hausdorff matrices with suitable $\mu_n$ [for details, see Hardy (**5**), Chapter XI].

A sequence $\{\mu_n\}$ is said to be *totally monotone* if

$$\Delta^p\mu_n \geqslant 0, \quad \text{for all } n, p = 0, 1, 2,.... \tag{8}$$

We say that $\mu_n$ is a *moment constant* generated by the function $\chi(t)$, if

$$\mu_n = \int_0^1 t^n \, d\chi(t), \tag{9}$$

where $\chi(t)$ is a function of bounded variation in $(0, 1)$ and without loss of generality we may assume that $\chi(0) = 0$. If, in addition, we have

$$\chi(+0) = \chi(0) = 0 \quad \text{and} \quad \chi(1) = 1 \tag{10}$$

(so that $\mu_0 = 1$), then $\mu_n$ is called a *regular moment constant*.

With these definitions, Hausdorff (**6**) has proved the following lemmas:

LEMMA 6. *In order that a real sequence $\{\mu_n\}$ should satisfy the following inequality*

$$\sup_n \sum_k \binom{n}{k} |\Delta^{n-k}\mu_k| < \infty \tag{11}$$

*it is necessary and sufficient that $\mu_n$ be a moment constant.*

LEMMA 7. *The Hausdorff matrix $\lambda \equiv (H, \mu_n)$ is a K-matrix if and only if $\mu_n$ is a moment constant and it is a T-matrix if and only if $\mu_n$ is a regular moment constant.*

The following two lemmas to which I refer in the sequel have been proved by me elsewhere [(**16**), Theorems **1**, **2**].

LEMMA 8.† *In order that a real sequence $\mu_n$ should satisfy the inequality*

$$\sup_n \sum_k \binom{k}{n} |\Delta^{k-n}\mu_{n+1}| < \infty, \tag{12}$$

*it is necessary and sufficient that $\mu_n$ be a moment constant.*

LEMMA 9. *The quasi-Hausdorff matrix $\lambda^* \equiv (H^*, \mu_n)$ is a $\delta$-matrix if and only if $\mu_n$ is a moment constant and it is an $\alpha_0$-matrix if and only if $\mu_n$ is a regular moment constant.*

We have now, from Lemmas 7 and 9,

LEMMA 10. *The Hausdorff matrix $\lambda \equiv (H, \mu_n)$ is a K-matrix or a T-matrix if and only if the quasi-Hausdorff matrix $\lambda^* \equiv (H^*, \mu_n)$ is a $\delta$-matrix or an $\alpha_0$-matrix.*

† I have since heard from Dr. Kuttner that he has given a direct proof of Lemma 8, with an appeal to Lemma 6, and that his proof does not depend on the theory of moment constants. I am grateful to him for communicating to me the results in his paper (**9**).

## 2. Hausdorff and quasi-Hausdorff methods

We start with the following theorem of Hardy [(**5**), Theorem 219].

THEOREM. *If $\mu_n$ is a regular moment constant, corresponding to the function $\chi(t)$, then the conditions*

$$\int_0^1 \frac{|d\chi|}{t} < \infty, \qquad \int_0^1 \frac{d\chi}{t} = 1 \tag{13}$$

*are necessary and sufficient for $\lambda^* \equiv (H^*, \mu_n)$ to be a $T$-matrix.*

I have pointed out in (**17**) that it is enough for $\mu_n$ in the above theorem to be a moment constant, the regularity of $\mu_n$ being superfluous. But for $\lambda^* \equiv (H^*, \mu_n)$ to be a $K$-matrix, it is necessary that

$$\sup_n \sum_k \binom{k}{n} |\Delta^{k-n}\mu_n| < \infty.$$

Putting $\mu_n = \nu_{n+1}$, we get from Lemma 8 that it is necessary that $\mu_n$ be a moment constant. Thus we have, indeed, the following

THEOREM 1. *The matrix $\lambda^* \equiv (H^*, \mu_n)$ is a $K$-matrix if and only if*

$$\text{(i) } \mu_n \textit{ is a moment constant} \quad \textit{and} \quad \text{(ii) } \int_0^1 \frac{|d\chi|}{t} < \infty.$$

*Note.* The conditions stated are equivalent to the condition that, if $\nu_n = \mu_{n-1}$ $(n \geqslant 1)$ and $\nu_0$ is arbitrary, then $\nu_n$ is a moment constant.

Now, for $\lambda^* \equiv (H^*, \mu_n)$ to be a $T$-matrix, we must have, as proved by Hardy (**5**), that $\int_0^1 \frac{d\chi}{t} = 1$, which, by a theorem for Steiltjes integrals [(**4**) 273, Theorem 14] implies that $\int_0^1 \frac{|d\chi|}{t}$ also exists, and therefore the theorem of Hardy's stated earlier can be reworded as follows:

HARDY'S THEOREM. *In order that $\lambda^* \equiv (H^*, \mu_n)$ may be a $T$-matrix, it is necessary and sufficient that*

$$\text{(i) } \mu_n \textit{ be a moment constant} \quad \textit{and} \quad \text{(ii) } \int_0^1 \frac{d\chi}{t} = 1.$$

For the Hausdorff transformation, Knopp and Lorentz (**8**) have proved that, if $\lambda \equiv (H, \mu_n)$ defines a conservative or regular sequence-to-sequence transformation, then it defines an absolutely conservative or absolutely regular sequence-to-sequence transformation. The same

result has been proved, independently and almost at the same time, by Hilda Morley (**15**). But it follows easily from Lemma 5 and the result of (**16**) that the converse result is also true, and thus the Hausdorff matrix $\lambda \equiv (H, \mu_n)$ defines an absolutely conservative or absolutely regular sequence-to-sequence transformation if and only if it defines a conservative or regular transformation of the same type. I have extended this result, in (**17**), to the quasi-Hausdorff series-to-series transformations. In the light of Theorem 1 and Hardy's theorem, we prove now

THEOREM 2. *The quasi-Hausdorff matrix* $\lambda^* \equiv (H^*, \mu_n)$ *defines an absolutely conservative (or absolutely regular) sequence-to-sequence transformation if and only if it defines a conservative (or regular) transformation of the same type.*

*Proof.* Taking $\lambda^* \equiv (H^*, \mu_n)$ we have

$$\lambda^*_{nk} = \binom{k}{n}\Delta^{k-n}\mu_n \quad (k \geqslant n), \qquad \lambda^*_{nk} = 0 \quad (k < n).$$

We see from the proof of Lemma 9 [(**16**), Theorem 2] that the series-to-series transformation corresponding to the sequence-to-sequence transformation defined by $(H^*, \mu_n)$ is given by the matrix $(H^*, \mu_{n-1})$ and now, applying Lemma 4, with $H \equiv (H^*, \mu_{n-1})$, we get that

$$\sum_n |h_{nk}| = \sum_n \binom{k}{n}|\Delta^{k-n}\mu_{n-1}|.$$

From Lemma 6, $\qquad \sup_k \sum_n \binom{k}{n}|\Delta^{k-n}\mu_{n-1}| < \infty,$

if and only if $\mu_n$ is a moment constant and $\displaystyle\int_0^1 \frac{|d\chi|}{t} < \infty.$

Also, when $\mu_n$ is a moment constant,

$$\sum_n h_{nk} = \sum_n \int_0^1 \binom{k}{n}(1-t)^{k-n}t^{n-1}\, d\chi(t) = \int_0^1 \frac{d\chi}{t},$$

and so we have, from Theorem 1 and Hardy's theorem, the required result.

Next, we shall find the necessary and sufficient conditions that the matrix $\lambda \equiv (H, \mu_n)$ may be a $\delta_0$-matrix or an $\alpha_0$-matrix. Let $\lambda \equiv (H, \mu_n)$ define a series-to-series transformation. Then, in the notations of Lemmas 2 and 3, and taking $H \equiv \lambda = (H, \mu_n)$, we have

$$h_{nk} = \lambda_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k \quad (n \geqslant k), \qquad h_{nk} = 0 \quad (n < k),$$

and consequently

$$g_{nk} = h_{0k}+h_{1k}+...+h_{nk} = \sum_{m=k}^{n} \binom{m}{k} \Delta^{m-k}\mu_k.$$

Therefore

$$\begin{aligned} g_{nk}-g_{n,k+1} &= \sum_{m=k}^{n} \binom{m}{k} \Delta^{m-k}\mu_k - \sum_{m=k+1}^{n} \binom{m}{k+1} \Delta^{m-k-1}\mu_{k+1} \\ &= \sum_{m=k}^{n} \left\{ \binom{m}{k} \Delta^{m-k}\mu_k + \binom{m}{k+1} \Delta^{m-k}\mu_k \right\} - \\ &\qquad - \sum_{m=k+1}^{n} \binom{m}{k+1} \{\Delta^{m-k}\mu_k + \Delta^{m-k-1}\mu_{k+1}\} \\ &= \sum_{m=k}^{n} \binom{m+1}{k+1} \Delta^{m-k}\mu_k - \sum_{m=k+1}^{n} \binom{m}{k+1} \Delta^{m-k-1}\mu_k \\ &= \binom{n+1}{k+1} \Delta^{n-k}\mu_k. \end{aligned} \tag{14}$$

Now for $(H, \mu_n)$ to be a $\delta_0$-matrix, we must have, by Lemmas 2 and 3,

$$\sup_n \sum_{k=0}^{n} \binom{n+1}{k+1} |\Delta^{n-k}\mu_k| < \infty. \tag{14 a}$$

Let, now, $\nu_n = \mu_{n-1}$ $(n \geqslant 1)$ and $\nu_0$ be arbitrary.

Then, by an obvious change of variable,

$$\sum_{k=0}^{n} \binom{n+1}{k+1} |\Delta^{n-k}\mu_k| = \sum_{k=1}^{n+1} \binom{n+1}{k} |\Delta^{n+1-k}\nu_k|.$$

But
$$\nu_0 = \sum_{k=0}^{n+1} \binom{n+1}{k} \Delta^{n+1-k}\nu_k,$$

so that
$$\Delta^{n+1}\nu_0 = \nu_0 - \sum_{k=1}^{n+1} \binom{n+1}{k} \Delta^{n+1-k}\nu_k,$$

whence

$$\sum_{k=0}^{n+1} \binom{n+1}{k} |\Delta^{n+1-k}\nu_k| \leqslant \nu_0 + 2\sum_{k=1}^{n+1} \binom{n+1}{k} |\Delta^{n+1-k}\nu_k|.$$

Thus (14 a) holds if and only if

$$\sup_n \sum_{k=0}^{n+1} \binom{n+1}{k} |\Delta^{n+1-k}\nu_k| < \infty.$$

By Lemma 6, this will hold if and only if $\nu_n$ is a moment constant. If this is satisfied, then, writing

$$\nu_n = \int_0^1 t^n \, d\alpha(t),$$

we have, for any fixed $k$,

$$g_{nk} = \sum_{m=k}^{n} \binom{m}{k} \Delta^{m-k} \nu_{k+1} = \int_0^1 \left\{ \sum_{m=k}^{n} \binom{m}{k} (1-t)^{m-k} t^{k+1} \right\} d\alpha(t) \to \int_{+0}^1 d\alpha(t)$$

as $n \to \infty$. Thus $\lambda$ is a $\delta_0$-matrix if and only if $\nu_n$ is a moment constant and an $\alpha_0$-matrix if and only if, in addition,

$$\int_{+0}^1 d\alpha(t) = 1,$$

which, as explained in the note to Theorem 1, gives, at once

THEOREM 3.† *The matrix* $\lambda \equiv (H, \mu_n)$ *is a* $\delta_0$*-matrix if and only if* (i) $\mu_n$ *is a moment constant and* (ii) $\int_0^1 \frac{|d\chi|}{t} < \infty$, *and it is an* $\alpha_0$*-matrix if and only if in addition* $\int_0^1 \frac{d\chi}{t} = 1.$

Thus we have from Theorems 1, 3, and Hardy's theorem a result exactly parallel to the one stated in Lemma 10, with the roles of Hausdorff and quasi-Hausdorff matrices interchanged.

To present a synoptic picture of the summability properties of these two methods, I prove also

THEOREM 4. *The Hausdorff matrix* $\lambda \equiv (H, \mu_n)$ *defines an absolutely conservative* (*or absolutely regular*) *series-to-series transformation if and only if it defines a conservative* (*or regular*) *transformation of the same type.*

The proof of the theorem is evident from Hardy's theorem and Theorem 1 and Lemma 6 since in the notation of Lemma 4, if we take $H \equiv \lambda = (H, \mu_n)$, then $h_{nk} \equiv \lambda_{nk}$, given by (6). The result in Theorem 4 may be placed in the same context as that in Theorem 2.

We have from Lemma 10 and the theorems proved so far that there exists between the $(H, \mu_n)$ and $(H^*, \mu_n)$ matrices a remarkably close relationship when we consider them as matrices defining summability methods. It is natural to examine the efficiency of these methods for

† My earlier proof of Theorem 3 was inadequate and the present proof was suggested to me by the referee. I am thankful to him for this and various other helpful suggestions.

summing sequences or series and also to compare the strengths of the two methods. These results are proved in the sequel.

Let us start with a matrix $(H, \mu_n)$. If this is a $T$-matrix, then we have, from Lemma 10, that $(H^*, \mu_n)$ is an $\alpha_0$-matrix and conversely. Also in the proof of Lemma 9 in (**16**), I have proved that $(H^*, \mu_{n+1})$ represents that $T$-matrix corresponding to the $\alpha_0$-matrix $(H^*, \mu_n)$. We now investigate the summability strengths of these two methods $(H, \mu_n)$ and $(H^*, \mu_{n+1})$. In the sequel, whenever we refer to $(H^*, \mu_{n+1})$ as a $T$-matrix, we, of course, take it as the $T$-matrix corresponding to the $\alpha_0$-matrix $(H^*, \mu_n)$.

To start with, we have the following lemma, due to Meyer-König [(**14**), Satz 25].

LEMMA 11. *If $s_n = O(1)$, then the methods $B$, $E(p)$ $(0 < p < 1)$, $T(\alpha)$ $(0 < \alpha < 1)$, and $S(\beta)$ $(0 < \beta < 1)$ are all equivalent.*

Here $B$ denotes the method of Borel's exponential means, $E(p)$ the Euler sequence-to-sequence method, defined by the matrix

$$\{E(p)\}_{nk} = \binom{n}{k}(1-p)^{n-k}p^k \quad (n \geqslant k); \quad = 0 \quad (n < k).$$

The method $T(\alpha)$ is the same as the method $F(1-\alpha)$ of Taylor series continuation, defined by Vermes (**18**) and also the 'circle method' $(\gamma, \alpha)$ of Hardy [(**5**) 219], while the method $S(\beta)$, defined by Meyer-König (**14**) is given by the matrix

$$\{S(\beta)\}_{nk} = \binom{n+k}{k}(1-\beta)^{n+1}\beta^k,$$

which is the same as the method $F(1-\beta)$ of Laurent series continuation, defined by Vermes (**18**).

Using Lemma 11, we prove now

THEOREM 5. *Let $s_n = O(1)$ and be summable by $B$ (Borel's exponential method) to $l$. Then every $T$-matrix $(H, \mu_n)$ sums $\{s_n\}$ to $l$ if the $\chi$-function generating the moment constants $\mu_n$ is continuous at $t = 1$.*

*Proof.* Let $\{t_n\}$ denote the transform of the sequence $\{s_n\}$ by the $(H, \mu_n)$-matrix. Then by hypothesis of the regularity of the method we have

$$\begin{aligned} t_n &= \sum_k \lambda_{nk} s_k \\ &= \sum_{k=0}^{n} \int_0^1 \binom{n}{k}(1-t)^{n-k}t^k s_k \, d\chi(t) \\ &= \int_0^1 \sum_{k=0}^{n} \binom{n}{k}(1-t)^{n-k}t^k s_k \, d\chi(t), \end{aligned}$$

where $\chi(t)$ satisfies the conditions

$$\chi(1) = \chi(1-0) = 1, \qquad \chi(0) = \chi(+0) = 0.$$

Denoting the integrand by $\{E_n(t)\}$ we have that $\{E_n(t)\}$ is the Euler transform, of order $t$, of $\{s_n\}$. But, by hypothesis of Borel summability of the sequence and by Lemma 11, we have that $E_n(t) \to l$ as $n \to \infty$ for every $t$ in $0 < t < 1$. This along with the continuity of $\chi(t)$ at $t = 0$ and $t = 1$ and the boundedness of

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} s_k$$

at both these points $t = 0$ and $t = 1$, gives that $t_n \to l$ and this completes the proof of the theorem.

Theorem 5 gives at once, as a special case, the following result for the Cesàro method $(C, k)$ $(k > 0)$ since the function $\chi(t)$ associated with the method $(C, k)$ is given by

$$\chi(t) = 1-(1-t)^k,$$

which obviously satisfies the conditions of Theorem 5.

THEOREM 6. *If $s_n = O(1)$ and is summable to $l$ by Borel's method, then it is summable $(C, \epsilon)$ to $l$ for every $\epsilon > 0$.*

*Note.* The above result is not essentially new, and in fact a more general result of the same kind has been proved by Hardy and Littlewood [see, for example, Hardy (**5**) 210, Theorem 147].

THEOREM 7. *Let $s_n = O(1)$ and be Borel summable to $l$. Then every $T$-matrix $(H^*, \mu_{n+1})$ is efficient for $\{s_n\}$ and sums it to $l$ provided that the function $\chi(t)$ generating the moment constants $\mu_n$ is continuous at $t = 1$ also.*

*Proof.* If $\{t_n^*\}$ is the transform of $\{s_n\}$ by the $(H^*, \mu_{n+1})$-matrix, then we have, by virtue of the regularity of the method, that

$$t_n^* = \sum_k \int_0^1 \binom{k}{n} (1-t)^{k-n} t^{n+1} s_k \, d\chi(t)$$

$$= \int_0^1 \sum_k \binom{k}{n} (1-t)^{k-n} t^{n+1} s_k \, d\chi(t),$$

the inversion of operations involved being justified by the boundedness of the sequence and the conditions on $\chi(t)$. Now the integrand is the $n$th term of the sequence obtained by the $T(1-t)$-transformation of the sequence $\{s_n\}$, and therefore we prove the theorem, as in Theorem 5, by appealing to Lemma 11.

Now we have immediately from Theorems 5 and 7, the theorem :

THEOREM 8. *Let $\{s_n\}$ be a bounded sequence which is Borel summable to $l$. Then the T-matrices $(H, \mu_n)$ and $(H^*, \mu_{n+1})$ are both efficient for the sequence $\{s_n\}$ and sum it to $l$ (i.e. they include Borel's method) provided that the function $\chi(t)$ which generates the moment constants $\mu_n$ is continuous at $t = 1$ also.*

*Remarks on Theorem* 8. It may be observed that we have assumed (i) that the sequence $\{s_n\}$ is bounded and Borel summable and (ii) that the regular transformations $(H, \mu_n)$ and $(H^*, \mu_{n+1})$ satisfy the additional hypothesis that the function $\chi(t)$ which generates the moment constants $\mu_n$ is continuous to the left at $t = 1$. That the condition (ii) above is also necessary for the validity of the theorems will be subsequently proved [see p. 210]. Now, it is natural to inquire whether the result will not be true for all sequences, i.e. without any restriction on them, and also whether the methods are not *totally equivalent*† for bounded sequences. I shall prove here that the methods are not necessarily totally equivalent for bounded sequences.

We have from Theorem 5.4.1 in Cooke [(**2**) 105] that for the methods $(H, \mu_n)$ and $(H^*, \mu_{n+1})$ to be totally equivalent for bounded sequences it is necessary and sufficient that

$$\sum_k \left| \binom{n}{k} \Delta^{n-k} \mu_k - \binom{k}{n} \Delta^{k-n} \mu_{n+1} \right| \to 0, \quad \text{as } n \to \infty. \tag{15}$$

Taking the methods to be positive, we have that (15) reduces to

$$\sum_{k=0}^{n-1} \binom{n}{k} \Delta^{n-k} \mu_k + (\mu_n - \mu_{n+1}) + \sum_{k=n+1}^{\infty} \binom{k}{n} \Delta^{k-n} \mu_{n+1} = 2(\mu_0 - \mu_{n+1}),$$

which cannot tend to zero unless $\mu_n \to 1$ since $\mu_0 = 1$.

Also that the methods are not equivalent for all sequences (without any restriction on them) is proved, by an example, by Kuttner. I am very grateful to Dr. Kuttner for this information.

Theorem 8 enables us to make the following remark on the Borel property of summability methods $(H, \mu_n)$. Hill (**7**) defines the notion of 'Borel property of summability methods' in the following way. The summability method $A$ is said to 'have the Borel property' if it sums almost all sequences of 0's and 1's to $\frac{1}{2}$. With this definition he proves that a regular Hausdorff method $(H, \mu_n)$ has the Borel property if and only if the function $\chi(t)$ associated with it is continuous at $t = 1$ also.

† Two summability methods A and B, transforming a sequence $\{s_n\}$ into $\{t_n\}$ and $\{t'_n\}$ respectively, are said to be *totally equivalent* if $\{t_n - t'_n\}$ always converges to zero, irrespective of the convergence of $\{t_n\}$ and $\{t'_n\}$. For further details on this, see Cooke (**2**).

A different proof of the same theorem was given also by Lorentz (**12**) at about the same time.

But it is known that the method of Borel's exponential means has the Borel property; also the sequences involved in discussing the Borel property are all bounded. Therefore it is evident that the sufficiency part of the Hill–Lorentz theorem is an immediate corollary of Theorem 8 above. The proof of the necessity, as provided by them, is simple.

## 3. Strong regularity

It will be convenient to start with the following definitions and known lemmas.

Following Lorentz (**10**), we say that a bounded sequence $\{s_n\}$ is *almost convergent* and write $S = \lim s_n$ if every Banach limit† of the sequence is $S$.

A method of summability which sums all almost convergent sequences is said to be *strongly regular*.

The *class* $\mathfrak{A}$ is defined as the entirety of the $T$-matrices $A \equiv (a_{nk})$ for which

$$\max_k |a_{nk}| \to 0 \quad \text{as } n \to \infty. \tag{16}$$

With these definitions, Lorentz (**10**, **11**) has proved the following results.

LEMMA 12.‡ *In order that the regular matrix method $A \equiv (a_{nk})$ (i.e. the $T$-matrix $A$) may be strongly regular, it is necessary and sufficient that*

$$\sum_k |a_{nk} - a_{n,k+1}| \to 0 \quad \textit{as } n \to \infty. \tag{17}$$

An immediate consequence of Lemma 12 and the definition of the class $\mathfrak{A}$ is that the class $\mathfrak{A}$ includes the class of strongly regular matrices, i.e. every strongly regular matrix is necessarily included in $\mathfrak{A}$.

LEMMA 13. *The $T$-matrix $A$ belongs to the class $\mathfrak{A}$ if and only if there exists a summability function of the first kind§ for the method.*

LEMMA 14. *A regular Hausdorff method $(H, \mu_n)$ is strongly regular if it belongs to the class $\mathfrak{A}$. For this the necessary and sufficient condition is that*

$$\mu_n \to 0 \quad \textit{as } n \to \infty, \tag{18}$$

*or, equivalently, that* $\chi(1) = \chi(1-0)$.

† For the definition of Banach limits, see Banach [(**1**) 33].

‡ This condition has been proved by Cooke (**3**) to be necessary and sufficient for the $T$-matrix $A$ to be 'absolutely regular (in his sense) for bounded sequences'. It is to be noted that Cooke [(**2**), chap. V or (**3**)] gives the term 'absolutely regular' a meaning completely different from that of the present paper.

§ For the definition of 'summability functions of the first kind', see Lorentz (**11**).

But, since, as stated earlier, the class $\mathfrak{A}$ includes the class of strongly regular methods, the above lemma in fact proves that the condition (18) above is necessary and sufficient for $(H, \mu_n)$ to be strongly regular or, equivalently, included in $\mathfrak{A}$.

From Lemma 14 we now pass on to determine the necessary and sufficient conditions for the strong regularity of the $T$-matrix $(H^*, \mu_{n+1})$ and also for the same to belong to $\mathfrak{A}$.

It has been proved by Lorentz (**10**) that every almost-convergent sequence (which is necessarily bounded also) is summable by the Euler's method $E(p)$ for $0 < p < 1$ and therefore certainly summable by Borel's method, by Lemma 11. Thus, if we know that the $T$-matrix $(H^*, \mu_{n+1})$ is such that the function $\chi(t)$ associated with it is continuous at $t = 1$, then, by Theorem 7, every almost-convergent sequence is summable $(H^*, \mu_{n+1})$, i.e. the matrix $(H^*, \mu_{n+1})$ is strongly regular if $\chi(1) = \chi(1-0)$.

That the condition is also necessary follows from Lemma 12 since, in case $\lambda^* \equiv (H^*, \mu_{n+1})$ is strongly regular, then $\mu_n \to 0$ because

$$\mu_{n+1} = \lambda^*_{nn} = \sum_{k=n}^{\infty} (\lambda^*_{nk} - \lambda^*_{n,k+1}) - \lim_{k\to\infty} \lambda^*_{nk} \leqslant \sum_{k=n}^{\infty} |\lambda^*_{nk} - \lambda^*_{n,k+1}| - \lim_{k\to\infty} \lambda^*_{nk},$$

$$\lim_{k\to\infty} \lambda^*_{nk} = 0$$

for each $n$ since the matrix $\lambda^*$ in question is, by hypothesis, a $T$-matrix. Thus we have proved the following theorem:

THEOREM 9. *The quasi-Hausdorff T-matrix* $(H^*, \mu_{n+1})$ *is strongly regular if and only if*

$$\mu_n \to 0 \quad \text{as } n \to \infty,$$

*or, equivalently, the function* $\chi(t)$ *associated with the method is continuous at* $t = 1$.

Using Theorem 9 we prove

THEOREM 10. *The quasi-Hausdorff T-matrix* $(H^*, \mu_{n+1})$ *belongs to the class* $\mathfrak{A}$ *under the same conditions as in Theorem* 9.

*Proof.* We have that, if

$$\lambda^* \equiv (H^*, \mu_{n+1}),$$

then

$$\lambda^*_{nn} \leqslant \max_k |\lambda^*_{nk}|,$$

and therefore $\lambda^*_{nn} = \mu_{n+1} \to 0$ whenever $\lambda^* \equiv (H^*, \mu_{n+1}) \subset \mathfrak{A}$. Also, when $\mu_n \to 0$, we have by the previous theorem that $(H^*, \mu_{n+1})$ is strongly regular and therefore necessarily belongs to $\mathfrak{A}$.

We are now in a position to prove the following theorem, which is the analogue of Lorentz's theorem for Hausdorff methods [(**11**) Theorem 9].

THEOREM 11. *A quasi-Hausdorff T-matrix $(H^*, \mu_{n+1})$ cannot be a gap method*† *and, if it is strongly regular, then all the functions $\Omega(n) = o(\sqrt{n})$ are summability functions of the first kind for the method $(H^*, \mu_{n+1})$.*

The proof of the first part of the theorem can be obtained by a straightforward adaptation of Lorentz's arguments in his theorem referred to above.

The second part follows from Theorems 8 and 10 and the result, due to Lorentz (**11**), that any function $\Omega(n) = o(\sqrt{n})$ is a summability function of the first kind for the Borel method.

It may, however, be observed that Theorems 9, 10, and the second part of Theorem 11 can also be proved directly, without reference to Theorem 8 and Lorentz's theorems, by proving first that the method $T(\alpha)$ defined earlier is strongly regular and also belongs to the class $\mathfrak{A}$ and that $\Omega(n) = o(\sqrt{n})$ are summability functions of the first kind for the method. But we choose the present approach to impress the close connexion between the Hausdorff and quasi-Hausdorff methods and also for the sake of brevity.

In the light of the results proved in this section, we are in a position to make the following remarks.

*Further remarks on Theorem* 8. In Theorem 8 it is also possible to prove that the continuity of $\chi(t)$ at $t = 1$ is necessary for the truth of the theorem, for, if the function is not continuous at $t = 1$, then we have, in virtue of Theorem 9 and that of Lorentz (**11**), quoted here as Lemma 14, that the methods $(H^*, \mu_{n+1})$ and $(H, \mu_n)$ are not strongly regular and therefore there exist almost-convergent sequences (which are necessarily bounded) not summable by the methods $(H^*, \mu_{n+1})$ and $(H, \mu_n)$. But every almost-convergent sequence is Borel summable. Thus we shall have that, if the continuity condition is not satisfied, then there exist bounded Borel summable sequences which are not summable by the methods $(H^*, \mu_{n+1})$ or $(H, \mu_n)$. Thus we have also that the conditions are necessary.

## 4. The $(S^*, \mu)$ method

In this section we define a class of methods and study briefly their relationship to the Hausdorff and quasi-Hausdorff methods.

The matrix $S^* \equiv (S^*, \mu)$ is defined as

$$s^*_{nk} = \binom{n+k}{k} \Delta^k \mu_{n+1} \quad (n,\, k = 0,\, 1,\, 2,\ldots). \tag{19}$$

† For definition, see Lorentz [(**11**) Theorem 8].

We shall find the necessary and sufficient conditions that $S^*$ may be a $K$-matrix or a $T$-matrix.

THEOREM 12. *The matrix $S^* \equiv (S^*, \mu)$ is a K-matrix if and only if $\mu_n$ is a moment constant; it is a T-matrix if and only if*

(i) *$\mu_n$ is a moment constant,*

(ii) $\displaystyle\int_{+0}^{1} d\chi(t) = 1,$

(iii) *the function $\chi(t)$ generating the $\{\mu_n\}$ is continuous at $t = 1$.*

*Proof.* In order that $S^*$ may be a $K$-matrix, we should have

$$\sup_n \sum_{k=0}^{\infty} |s^*_{nk}| = \sup_n \sum_{k=0}^{\infty} \binom{n+k}{k} |\Delta^k \mu_{n+1}| < \infty. \tag{20}$$

But
$$\sum_{k=0}^{\infty} \binom{n+k}{k} |\Delta^k \mu_{n+1}| = \sum_{k'=n+k=n}^{\infty} \binom{k'}{n} |\Delta^{k'-n} \mu_{n+1}|,$$

and it therefore follows from Lemma 8 that, in order that (20) should hold, it is necessary and sufficient that $\mu_n$ be a moment constant. Also, when $\mu_n$ is a moment constant,

$$\sum_{k=0}^{\infty} s^*_{nk} = \sum_{k=0}^{\infty} \int_0^1 \binom{n+k}{k} (1-t)^k t^{n+1}\, d\chi(t) = \int_{+0}^{1} d\chi(t).$$

Further,
$$\lim_{n\to\infty} s^*_{nk} = \delta_k,$$

which, as in the proof of Lemma 6, exists for each $k$ and is in fact zero when $k \geqslant 1$. Thus we have the first part of the theorem.

For the proof of the second part, we have only to prove the necessity of condition (iii). Now, when $k = 0$,

$$\lim_{n\to\infty} s^*_{nk} = \lim_{n\to\infty} \mu_{n+1}$$

and thus the above limit is zero if and only if $\chi(t)$ is continuous at $t = 1$, as proved by Lorentz [(**10**) Theorem 13].

*Note* 1. The method $S(\alpha)$ of Meyer-König (**14**), called also the 'method $F(\alpha)$' of Laurent-series continuation by Vermes (**18**), is a special case of the $(S^*, \mu)$ method, with $\mu_n = (1-\alpha)^n$. Consequently, we have that the function $\chi(t)$ is defined by

$$\chi(t) = \begin{cases} 0 & (0 < t < 1-\alpha), \\ 1-\alpha & (1-\alpha \leqslant t \leqslant 1), \end{cases}$$

which satisfies the conditions of the above theorem, and therefore the

method is regular if and only if $0 < \alpha < 1$, as proved by Meyer-König (**14**) and Vermes (**18**).

*Note* 2. Now, from Lemma 12, the $T$-method $(S^*, \mu)$ is strongly regular if and only if

$$\sum_{k=0}^{\infty} \left| \binom{n+k}{k} \Delta^k \mu_{n+1} - \binom{n+k+1}{k+1} \Delta^{k+1} \mu_{n+1} \right| \to 0 \quad \text{as } n \to \infty,$$

i.e. if and only if

$$\sum_{k'=n}^{\infty} \left| \binom{k'}{n} \Delta^{k'-n} \mu_{n+1} - \binom{k'+1}{n} \Delta^{k'+1-n} \mu_{n+1} \right| \to 0 \quad \text{as } n \to \infty.$$

But, if $\mu_n$ is a moment constant, then the above equation is true if and only if $\mu_n \to 0$, by virtue of Theorem 9. Thus we have that, *if the method* $(S^*, \mu)$ *is regular, then it must necessarily be strongly regular.*

Finally, by appealing to Lemma 11 as in the proof of Theorems 5 and 7, we can prove the following theorem:

THEOREM 13. *If* $s_n = O(1)$ *and is summable by Borel's method to* $l$, *then the* $T$-*method* $(S^*, \mu)$ *is efficient for* $\{s_n\}$ *and includes the Borel method.*

For we have

$$\sum_{k=0}^{N} \binom{n+k}{k} \Delta^k \mu_{n+1} s_k = \int_0^1 \left\{ t^{n+1} \sum_{k=0}^{N} \binom{n+k}{k} (1-t)^k s_k \right\} d\chi(t)$$

$$= \int_{+0}^1 \left\{ t^{n+1} \sum_{k=0}^{N} \binom{n+k}{k} (1-t)^k s_k \right\} d\chi(t)$$

since the expression inside the brackets $\{\}$ vanishes when $t = 0$. The inversion in the order of integration and the limit $N \to \infty$ is easily justifiable, and therefore we have the theorem, arguing as in the proofs of Theorems 5 and 7.

Thus we see that, *whenever the function* $\chi(t)$ *associated with the moment constants* $\mu_n$ *is continuous at* $t = 1$, *then the* $T$-*methods* $(H, \mu_n)$, $(H^*, \mu_{n+1})$, *and* $(S^*, \mu)$ *all include Borel's method for bounded sequences.*

In conclusion I have great pleasure in thanking Professor C. T. Rajagopal and Professor V. Ganapathy Iyer for their help and guidance in the preparation of this paper. I am greatly indebted to Dr. B. Kuttner for his scrutiny of my manuscript and for his very valuable suggestions on various points.

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# ON QUARTIC FIELDS OF SIGNATURE ONE WITH SMALL DISCRIMINANT

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**1.** IN two recent papers (**2**), (**3**), I have developed and applied a method for the enumeration of all the algebraic fields of given degree and signature which have no subfields and have discriminants numerically less than some given bound. In the present paper this method is applied to the remaining case of quartic fields—those with signature one. For the sake of completeness a table of the fields with subfields is also included, so that all quartic fields with discriminant $\Delta$ satisfying $-3280 \leqslant \Delta < 0$ are enumerated. We thus have an extension of the table given for $-848 \leqslant \Delta < 0$ by Delone and Faddeev (**1**).

**2.** The basis of the method is the following theorem.

THEOREM. *Let $K$ be a quartic field with no subfield having signature one and discriminant $\Delta$. Then there is at least one polynomial*

$$f(x) = x^4 - ax^3 + bx^2 - cx + d$$

*with zeros $\alpha$, $\beta$, $\gamma \pm i\delta$ ($\alpha$, $\beta$, $\gamma$, $\delta$ real) for which*

$$S = (\alpha-\beta)^2 + (\gamma-\alpha)^2 + (\gamma-\beta)^2 + \sigma\delta^2 \leqslant (-3\sigma\Delta/2)^{\frac{1}{2}} \quad (\sigma > 0)$$

*such that $K$ is generated by one of the zeros of $f(x)$.*

The proof of this theorem is the same as that of the theorem in (**3**), except that the matrix

$$J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{pmatrix} \text{ is replaced by } \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -i\sqrt{(\frac{1}{12}\sigma)} & i\sqrt{(\frac{1}{12}\sigma)} \end{pmatrix}.$$

**3.** Since we are concerned ultimately with the field generated by a zero of $f(x)$, we may add any rational integer on to the zeros of $f(x)$, or subtract them from any rational integer, or permute them. We may therefore suppose whenever necessary that

$$0 \leqslant \alpha < 1, \qquad \alpha \leqslant \beta, \qquad \alpha+\beta \leqslant 2\gamma,$$

and this gives

$$4ab - a^3 - 8c = (2\gamma-\alpha-\beta)\{(\alpha-\beta)^2 + 4\delta^2\} \geqslant 0. \tag{1}$$

Since equality in any one of these would mean that $f(x)$ was reducible in a field of degree less than four, we may suppose that we have strict inequality.

4. The choice of a numerical value for $\sigma$ will be discussed in § 5. In this section we establish some properties of $S$ which are independent of a particular choice of $\sigma$. $S$ satisfies a sextic equation with coefficients which are polynomials in $a$, $b$, $c$, $d$ and is best dealt with indirectly: this is in contrast to the situation obtaining in the other types of quartic field, for in totally real fields the expression analogous to $S$ is a simple function of $a$ and $b$, while in totally complex fields the analogous expression satisfies a cubic equation which can be fairly easily used to obtain bounds on $a$, $b$, $c$, $d$.

I state as lemmas some properties of $S$ which will be used later.

LEMMA 1. *For $a$, $b$, $c$ given, $S$, regarded as a function of $d$, is monotonic on either side of a single minimum, and its value at this minimum is*

$$\tfrac{1}{16}[3(\sigma+2)\{(\sigma+6)(4ab-a^3-8c)/(\sigma+2)\}^{\frac{2}{3}}+(6-\sigma)(3a^2-8b)].$$

*Proof.* There may, or may not, be values of $d$ for which $f(x)$ has four real zeros: suppose first that there are not. Then $f(x)$ has just two real zeros for $d < d_0$ and none for $d_0 < d$. Now

$$S = -\tfrac{1}{8}\sigma(3a^2-8b)+\tfrac{1}{2}(\sigma+4)(\tfrac{1}{2}a-\alpha-\beta)^2+\tfrac{1}{4}(\sigma+6)(\alpha-\beta)^2,$$

$$\frac{\partial\alpha}{\partial d} = -\frac{1}{f'(\alpha)} = \frac{1}{(\beta-\alpha)\{(\alpha-\gamma)^2+\delta^2\}},$$

$$\frac{\partial\beta}{\partial d} = -\frac{1}{f'(\beta)} = \frac{1}{(\alpha-\beta)\{(\beta-\gamma)^2+\delta^2\}}.$$

Hence

$$\{(\alpha-\gamma)^2+\delta^2\}\{(\beta-\gamma)^2+\delta^2\}\frac{\partial S}{\partial d}$$

$$= \tfrac{1}{4}(\sigma+2)(2\gamma-\alpha-\beta)^2-\tfrac{1}{4}(\sigma+6)(\alpha-\beta)^2-(\sigma+6)\delta^2$$

$$= \tfrac{1}{4}(\sigma+2)(2\gamma-\alpha-\beta)^2-\tfrac{1}{4}(\sigma+6)(4ab-a^3-8c)/(2\gamma-\alpha-\beta).$$

Now $u = 2\gamma-\alpha-\beta$ satisfies the equation

$$u^6-(3a^2-8b)u^4+(3a^4-16a^2b+16b^2+16ac-64d)u^2-$$

$$-(4ab-a^3-8c)^2 = 0.$$

An increase in $d$ increases $2\gamma-\alpha-\beta$ and so, if $\partial S/\partial d$ is positive for some value of $d$, it is positive for all larger values, while, if it is negative for some $d$, it is negative for all smaller values. Thus $S$ varies monotonically on either side of a minimum value as stated.

At this minimum we have

$$-(\sigma+6)\delta^2-\tfrac{1}{4}(\sigma+6)(\alpha-\beta)^2+\tfrac{1}{4}(\sigma+2)(2\gamma-\alpha-\beta)^2 = 0,$$
$$\sigma\delta^2+\tfrac{3}{2}(\alpha-\beta)^2+\tfrac{1}{2}(2\gamma-\alpha-\beta)^2 = S,$$
$$-8\delta^2+2(\alpha-\beta)^2+(2\gamma-\alpha-\beta)^2 = 3a^2-8b.$$

Solving for $\delta^2$, $(\alpha-\beta)^2$, $(2\gamma-\alpha-\beta)^2$ and substituting in

$$\begin{aligned}c &= 2\alpha\beta\gamma+(\alpha+\beta)(\gamma^2+\delta^2)\\ &= \tfrac{1}{2}\delta^2\{a-(2\gamma-\alpha-\beta)\}-\tfrac{1}{8}(\alpha-\beta)^2(a+2\gamma-\alpha-\beta)+\tfrac{1}{16}a\{a^2-(2\gamma-\alpha-\beta)^2\},\end{aligned}$$

we obtain

$$c = \tfrac{1}{16}a^3-\tfrac{1}{16}a(3a^2-8b)-\tfrac{1}{8}(\sigma+2)(2\gamma-\alpha-\beta)^3/(\sigma+6),$$

whence the stated result follows.

The minimum may be unattained since the value obtained for $\delta^2$ may be negative.

If $f(x)$ does have four real zeros for certain values of $d$, then an ambiguity in the definition of $S$ is introduced. If $f(x)$ has two real zeros for $d_1 < d < d_0$ (region I) and $d < d_2$ (region III) and four real zeros for $d_2 < d < d_1$ (region II), then the real zeros in (I) correspond to the two least zeros in (II), but the real zeros in (III) correspond to the least and greatest zeros in (II). Thus in going from region I to region III we must at some stage change the expression used for $S$. We can agree to use the same definition $S_{\mathrm{I}}$ of $S$ in regions I and II and change to the other definition $S_{\mathrm{III}}$ when $d = d_2$ and $f(x)$ has zeros $\alpha_0$, $\beta_0$, $\beta_0$, $\delta_0$.

Then, for $d = d_2$,

$$\begin{aligned}S_{\mathrm{III}}-S_{\mathrm{I}} &= \{(\alpha_0-\beta_0)^2+(\alpha_0-\delta_0)^2+(\delta_0-\beta_0)^2\}-\\ &\quad -\{(\alpha_0-\beta_0)^2+(\alpha_0-\tfrac{1}{2}(\beta_0+\delta_0))^2+(\beta_0-\tfrac{1}{2}(\beta_0+\delta_0))^2-\tfrac{1}{4}\sigma(\beta_0-\delta_0)^2\}\\ &= \tfrac{3}{4}(\delta_0-\beta_0)^2+\tfrac{1}{2}(\beta_0-\delta_0)(2\alpha_0-\tfrac{1}{2}\beta_0-\tfrac{3}{2}\delta_0)+\tfrac{1}{4}\sigma(\beta_0-\delta_0)^2\\ &> \tfrac{1}{4}(\delta_0-\beta_0)(6\delta_0-2\beta_0-4\alpha_0) > 0.\end{aligned}$$

Also, at $d = d_2$, $\partial S_{\mathrm{III}}/\partial d$ has the sign of

$$\begin{aligned}&\tfrac{1}{4}(\sigma+2)(2\beta_0-\alpha_0-\delta_0)^2-\tfrac{1}{4}(\sigma+6)(\delta_0-\alpha_0)^2\\ &\qquad < \tfrac{1}{4}(\sigma+2)(\delta_0-\alpha_0)^2-\tfrac{1}{4}(\sigma+6)(\delta_0-\alpha_0)^2 < 0\end{aligned}$$

since $2\beta_0 > \alpha_0+\delta_0$, $\delta_0 > \beta_0$.

Thus no further minimum is introduced by the change of definition of $S$, and this completes the proof of the lemma.

LEMMA 2. *$S \geqslant S^* = -\tfrac{1}{8}\sigma(3a^2-8b)+\tfrac{1}{2}(\sigma+4)(\tfrac{1}{2}a-\alpha-\beta)^2$. For $a$, $b$ fixed, $S^*$ is an increasing function of $d$ and a decreasing function of $c$.*

*Proof.* $$S-S^* = \tfrac{1}{4}(\sigma+6)(\alpha-\beta)^2.$$

Also, since $$\frac{\partial\alpha}{\partial c} = \frac{\alpha}{f'(\alpha)}, \qquad \frac{\partial\beta}{\partial c} = \frac{\beta}{f'(\beta)},$$

we have $$\frac{\partial}{\partial c}(\alpha+\beta) = \frac{\delta^2+\gamma^2-\alpha\beta}{\{(\gamma-\alpha)^2+\delta^2\}\{(\gamma-\beta)^2+\delta^2\}} > 0,$$

while $$\frac{\partial}{\partial d}(\alpha+\beta) = \frac{\beta+\alpha-2\gamma}{\{(\gamma-\alpha)^2+\delta^2\}\{(\gamma-\beta)^2+\delta^2\}} < 0.$$

LEMMA 3. $S > \frac{3}{2}(\beta-\alpha)^2$. *Also, for $a$, $b$, $\beta$ fixed, $S$ is a decreasing function of $\alpha$.*

*Proof.* $$S-\tfrac{3}{2}(\beta-\alpha)^2 = \tfrac{1}{2}(2\gamma-\alpha-\beta)^2+\sigma\delta^2.$$

Since
$$S = -\tfrac{1}{8}\sigma(3a^2-8b)+\tfrac{1}{2}(\sigma+4)(\tfrac{1}{2}a-\alpha-\beta)^2+\tfrac{1}{4}(\sigma+6)(\alpha-\beta)^2,$$

we have
$$\frac{\partial S}{\partial\alpha} = -(\sigma+4)(\tfrac{1}{2}a-\alpha-\beta)+\tfrac{1}{2}(\sigma+6)(\alpha-\beta)$$
$$= -(\sigma+4)(\gamma-\beta)-(\sigma+5)(\beta-\alpha) < 0.$$

LEMMA 4. $a < 4+\sqrt{(14S/3)}$.

*Proof.* For given $S$, $a$ will be greatest when $\alpha$ is as large as possible and $\delta = 0$. We thus have to maximize $a = 1+\beta+2\gamma$ subject to
$$S = (\beta-1)^2+(\gamma-1)^2+(\gamma-\beta)^2$$
(the bound for $a$ being unattained since $\alpha < 1$). By elementary calculus we find that the maximum occurs for
$$\beta-1 = \tfrac{4}{5}(\gamma-1) = \sqrt{(16S/42)},$$
and this gives $a = 4+\sqrt{(14S/3)}$.

5. We now discuss the choice of a numerical value for $\sigma$. In applying the method we choose $\Delta_0$ such that we wish to find all fields with discriminant not less than $\Delta_0$: having chosen $\Delta_0$ the value $S_0 = (-3\sigma\Delta_0/2)^{\frac{1}{3}}$ is obtained from the theorem in § 2, and we consider all polynomials with $S \leqslant S_0$. These polynomials will have discriminants greater than or equal to $\Delta_1$. The criterion which we employ in choosing $\sigma$ is that $|\Delta_1|$ should be as small as possible: we may hope that this will maximize the proportion of polynomials considered for which the zeros yield fields with discriminants not less than $\Delta_0$, but in view of the imprecise relationship between the discriminants of a polynomial and of the fields generated by its zeros, this can be no more than a vague hope.

To find $\Delta_1$ we maximize

$$\surd(-\Delta) = 2(\beta-\alpha)\{(\gamma-\alpha)^2+\delta^2\}\{(\gamma-\beta)^2+\delta^2\}\delta$$

subject to $\quad S_0 = (\beta-\alpha)^2+(\gamma-\alpha)^2+(\gamma-\beta)^2+\sigma\delta^2.$

To simplify the working we may take $\gamma = 0$ and then, using the method of Lagrange's multipliers, we consider

$$2(\beta-\alpha)(\alpha^2+\delta^2)(\beta^2+\delta^2)\delta-\lambda\{S_0-(\beta-\alpha)^2-\alpha^2-\beta^2-\sigma\delta^2\}$$

and have

$$-2(\alpha^2+\delta^2)(\beta^2+\delta^2)\delta+4\alpha(\beta-\alpha)(\beta^2+\delta^2)\delta+2\lambda(2\alpha-\beta) = 0, \tag{2}$$

$$2(\alpha^2+\delta^2)(\beta^2+\delta^2)\delta+4\beta(\beta-\alpha)(\alpha^2+\delta^2)\delta+2\lambda(-\alpha+2\beta) = 0, \tag{3}$$

$$2(\beta-\alpha)\{5\delta^4+3\delta^2(\alpha^2+\beta^2)+\alpha^2\beta^2\}+2\lambda\sigma\delta = 0. \tag{4}$$

Adding (2) and (3) we have

$$(\alpha+\beta)\{4\delta(\beta-\alpha)(\alpha\beta+\delta^2)+2\lambda\} = 0. \tag{5}$$

Suppose first that

$$\lambda = -2\delta(\beta-\alpha)(\alpha\beta+\delta^2).$$

Then from (4) and (2), since we can exclude the possibilities $\delta = 0$ or $\beta = \alpha$, we have

$$\delta^4(5-2\sigma)+\delta^2(3\alpha^2-2\sigma\alpha\beta+3\beta^2)+\alpha^2\beta^2 = 0, \tag{6}$$

$$\delta^4-\delta^2(\alpha^2-4\alpha\beta+\beta^2)-(4\alpha^2-9\alpha\beta+4\beta^2)\alpha\beta = 0. \tag{7}$$

If $\beta = 0$, then $\delta^2(5-2\sigma)+3\alpha^2 = 0$ and also $\delta^2 = \alpha^2$, whence $\sigma = 4$; if $\alpha = 0$, then also $\beta = 0$ since $\alpha \leqslant \beta \leqslant 2\gamma-\alpha$, but we have excluded $\beta = \alpha$.

If neither of $\alpha$ or $\beta$ is 0, we put

$$\delta^2 = t\alpha\beta, \qquad \alpha\beta^{-1}+\beta\alpha^{-1} = u \tag{8}$$

and write (6) as $\quad (5-2\sigma)t^2+t(3u-2\sigma)+1 = 0,$

and (7) as $\quad t^2-t(u-4)-(4u-9) = 0.$

Eliminating $t$ we have

$$\{-12u^2+(28+8\sigma)u-(18\sigma+4)\}\{-(8-2\sigma)u+(20-6\sigma)\}$$
$$= \{-4(5-2\sigma)u+44-18\sigma\}^2,$$

i.e.

$$(u-2)\{(8-2\sigma)u^2-(4\sigma^2-28\sigma+56)u+(9\sigma^2-52\sigma+84)\} = 0. \tag{9}$$

The root $u = 2$ corresponds to $\beta = \alpha$ and is to be disregarded. Then (9) gives real roots for $u$ only if

$$(\sigma-2)^3(2\sigma-7) \geqslant 0,$$

i.e. if $\sigma \leqslant 2$ or $\frac{7}{2} \leqslant \sigma$.

If, in (5), $\alpha+\beta = 0$, then (2) gives

$$-2\delta(\alpha^2+\delta^2)(5\alpha^2+\delta^2)+6\lambda\alpha = 0, \tag{10}$$

and (4) gives
$$-4\alpha(\alpha^2+\delta^2)(\alpha^2+5\delta^2)+2\lambda\sigma\delta = 0, \tag{11}$$

whence
$$\sigma\delta^4+\alpha^2\delta^2(5\sigma-30)-6\alpha^4 = 0. \tag{12}$$

For given $\sigma$ we may now solve for $u$ from (9), then for $\beta/\alpha$ from (8), and then for $\alpha/\delta$, $\beta/\delta$ from (6) or (7), or else for $\alpha^2/\delta^2$ from (12). Having $\alpha/\delta$, $\beta/\delta$ we can find $\Delta_1$ in terms of $\Delta_0$.

For example, if $\sigma = 4$, then (9) gives $u = \frac{5}{2}$, whence $\beta = \frac{1}{2}\alpha$ (we have $\alpha \leqslant \beta \leqslant -\alpha$ when $\gamma = 0$).

Then (6) gives $\alpha^2 = 4\delta^2$, so that, taking $\alpha = -2\delta$, $\beta = -\delta$, we have

$$\sqrt{(-\Delta)} = 20\delta^6, \qquad S_0 = 10\delta^2,$$

whence
$$\sqrt{(-\Delta)} = S_0^3/50.$$

Again (12) gives $\alpha^2 = \frac{1}{3}\delta^2$, so that

$$\alpha = -\delta/\sqrt{3}, \qquad \beta = \delta/\sqrt{3}$$

and
$$\sqrt{(-\Delta)} = 64\delta^6/9\sqrt{3}, \qquad S_0 = 6\delta^2,$$

whence
$$\sqrt{(-\Delta)} = 8S_0^3/243\sqrt{3} < S_0^3/50.$$

For $\sigma = 4$ we have also the possibility $\alpha = -\delta$, $\beta = 0$ giving

$$\sqrt{(-\Delta)} = 4\delta^6, \qquad S_0 = 6\delta^2,$$

whence
$$\sqrt{(-\Delta)} = S_0^3/54 < S_0^3/50.$$

Thus $-\Delta_1 = S_0^6/2500 = 9\Delta_0^2/625$ since $-\Delta_0 = S_0^3/6$.

Calculation of $\Delta_1$ for other values of $\sigma$ shows that $\sqrt{(-\Delta_1)}/(-\Delta_0)$ has least value 0·118... for $\sigma$ approximately 4·6. Since it is convenient to take $\sigma$ to be an integer and since 0·12 is very little larger than 0·118, I shall take $\sigma = 4$ in the remainder of the work.

**6.** In this section we discuss bounds for $a, b, c, d$. All these are positive in virtue of the assumptions in § 3, and an upper bound for $a$ was given in Lemma 4. For $b$ we have

LEMMA 5. (i) $\frac{3}{8}a^2-\frac{1}{4}S < b$,

(ii) $b < \frac{1}{4}S+\frac{3}{8}a^2$,

(iii) $b < \frac{1}{4}S+(29a^2+80a-160)/104$ *for* $4 \leqslant a$.

*Proof.* (i) Since $\delta^2 > 0$, we have

$$S > (\alpha-\beta)^2+(\gamma-\alpha)^2+(\gamma-\beta)^2$$
$$= 2(\tfrac{1}{2}a-\alpha-\beta)^2+\tfrac{3}{2}(\alpha-\beta)^2 > 2(\tfrac{1}{2}a-\alpha-\beta)^2.$$

Also

$$S > (\alpha-\beta)^2+(\gamma-\alpha)^2+(\gamma-\beta)^2-6\delta^2 = \tfrac{3}{4}(3a^2-8b)-(\tfrac{1}{2}a-\alpha-\beta)^2.$$

Thus we have

$$\tfrac{3}{4}(3a^2-8b)-S < (\tfrac{1}{2}a-\alpha-\beta)^2 < \tfrac{1}{2}S,$$

whence the result follows.

(ii) $S = -\tfrac{1}{2}(3a^2-8b)+4(\tfrac{1}{2}a-\alpha-\beta)^2+\tfrac{5}{2}(\alpha-\beta)^2 > -\tfrac{1}{2}(3a^2-8b),$

whence the result follows.

(iii) $$S = -\tfrac{1}{2}(3a^2-8b)+a^2-4a(\alpha+\beta)+\tfrac{13}{2}(\alpha+\beta)^2-10\alpha\beta.$$

For all $\alpha+\beta$ we have $\alpha\beta < \tfrac{1}{4}(\alpha+\beta)^2$ while, if $2 \leqslant \alpha+\beta$, we have

$$\alpha\beta < (\alpha+\beta-1) \quad \text{since } \alpha < 1.$$

Hence, if $\alpha+\beta \leqslant 2$, then

$$S > -\tfrac{1}{2}(3a^2-8b)+4(\tfrac{1}{2}a-\alpha-\beta)^2 \geqslant -\tfrac{1}{2}(3a^2-8b)+4(\tfrac{1}{2}a-2)^2.$$

If $\alpha+\beta \geqslant 2$, then

$$S > -\tfrac{1}{2}(3a^2-8b)+a^2-4a(\alpha+\beta)+\tfrac{13}{2}(\alpha+\beta)^2-10(\alpha+\beta-1)$$

and, for variation in $\alpha+\beta$, this has minimum value

$$-\tfrac{1}{2}(3a^2-8b)+a^2+10-(4a+10)^2/26$$

given by $$\alpha+\beta = (4a+10)/13 \quad (\geqslant 2).$$

Since

$$4(\tfrac{1}{2}a-2)^2-\{a^2+10-(4a+10)^2/26\} = 8(a-4)^2/13 \geqslant 0,$$

then we certainly have that

$$S > -\tfrac{1}{2}(3a^2-8b)+a^2+10-(4a+10)^2/26,$$

whence the result follows.

Bounds on $c$ follow from (1) and Lemma 1.

Bounds on $d$ are not given explicitly but follow implicitly from Lemma 1 when we specify an upper bound for $S$. Lemmas 2 and 3 help to estimate $S$ easily, and Lemma 1 shows that the appropriate values of $d$ lie in just one interval. Nevertheless the actual determination of the values of $d$ for given $a$, $b$, $c$ is largely a matter of trial and error.

**7.** When $a$, $b$, $c$, $d$ have been tabulated the processes of determining the discriminants of the fields generated and investigating the equivalence of fields with the same discriminant follow as in (**2**) and (**3**). It is to be noted that, if a polynomial with discriminant $\Delta$ gives a field with discriminant $\Delta/k^2$, then we may suppose that $S \leqslant (-6\Delta/k^2)^{\frac{1}{3}}$ (since otherwise there will be some other polynomial for which this is true and the zeros of which generate the field) and we have from § 5 that

$-\Delta \leqslant S^6/2500$. Hence $S^3 \leqslant 6S^6/2500k^2$, i.e. $k^2 \leqslant 0{\cdot}0024S^3$, which gives a useful bound for $k$.

By taking $S = 27$ we obtain all fields with discriminants not less than $-\frac{1}{2}3^8 = -3280{\cdot}5$. The fields are generated by the zeros $\theta$ of $f(x)$, where $a$, $b$, $c$, $d$ are as given in Table 1: in every case except one $1, \theta, \theta^2, \theta^3$ provide a basis for the field, the exception being the field with discriminant $-2787$, for which $1, \theta, \theta^2, \frac{1}{3}(\theta^3+\theta^2)$ provide a basis.

TABLE 1

| Δ | a | b | c | d | Δ | a | b | c | d | Δ | a | b | c | d |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| −283 | 3 | 7 | 5 | 1 | −1856 | 10 | 35 | 46 | 11 | −2608 | 8 | 22 | 22 | 2 |
| −331 | 5 | 10 | 6 | 1 | −1879 | 11 | 43 | 66 | 28 | −2619 | 9 | 27 | 29 | 9 |
| −491 | 7 | 15 | 9 | 1 | −1927 | 9 | 28 | 31 | 4 | −2687 | 11 | 42 | 59 | 14 |
| −563 | 6 | 12 | 7 | 1 | −1931 | 8 | 24 | 29 | 11 | −2696 | 7 | 18 | 17 | 3 |
| −643 | 7 | 18 | 18 | 5 | −1963 | 7 | 20 | 26 | 11 | −2736 | 6 | 14 | 12 | 1 |
| −688 | 4 | 10 | 6 | 1 | −1968 | 6 | 14 | 10 | 2 | −2763 | 5 | 11 | 7 | 1 |
| −731 | 5 | 11 | 11 | 3 | −1984 | 6 | 14 | 12 | 2 | −2764 | 7 | 17 | 11 | 2 |
| −751 | 6 | 13 | 11 | 1 | −2051 | 9 | 29 | 37 | 15 | −2767 | 7 | 18 | 17 | 4 |
| −848 | 8 | 23 | 26 | 9 | −2068 | 9 | 28 | 33 | 11 | −2787 | 5 | 15 | 21 | 9 |
| −976 | 6 | 15 | 16 | 5 | −2092 | 9 | 27 | 29 | 8 | −2816 | 12 | 50 | 80 | 31 |
| −1099 | 7 | 19 | 21 | 7 | −2096 | 8 | 22 | 22 | 6 | −2824 | 7 | 17 | 14 | 2 |
| −1107 | 7 | 18 | 18 | 3 | −2116 | 5 | 10 | 9 | 1 | −2843 | 5 | 11 | 9 | 1 |
| −1192 | 5 | 11 | 10 | 2 | −2151 | 8 | 23 | 25 | 4 | −2859 | 9 | 29 | 37 | 11 |
| −1255 | 7 | 15 | 8 | 1 | −2183 | 6 | 15 | 15 | 4 | −2911 | 8 | 25 | 35 | 16 |
| −1328 | 4 | 9 | 6 | 1 | −2191 | 5 | 12 | 13 | 4 | −2943 | 7 | 18 | 19 | 4 |
| −1371 | 4 | 8 | 7 | 1 | −2219 | 5 | 12 | 14 | 5 | −3052 | 7 | 19 | 23 | 8 |
| −1399 | 7 | 16 | 11 | 2 | −2243 | 9 | 28 | 32 | 5 | −3119 | 3 | 9 | 6 | 1 |
| −1423 | 7 | 19 | 22 | 7 | −2284 | 9 | 26 | 24 | 2 | −3163 | 10 | 34 | 41 | 7 |
| −1424 | 8 | 25 | 34 | 15 | −2319 | 9 | 28 | 33 | 12 | −3175 | 8 | 23 | 25 | 8 |
| −1456 | 8 | 22 | 22 | 5 | −2327 | 8 | 23 | 27 | 8 | −3188 | 7 | 19 | 20 | 6 |
| −1472 | 6 | 14 | 14 | 3 | −2412 | 9 | 29 | 39 | 16 | −3216 | 10 | 35 | 48 | 21 |
| −1588 | 3 | 9 | 8 | 2 | −2443 | 8 | 24 | 29 | 9 | −3223 | 9 | 29 | 36 | 11 |
| −1732 | 9 | 27 | 28 | 2 | −2480 | 8 | 24 | 30 | 10 | −3267 | 5 | 12 | 12 | 3 |
| −1791 | 7 | 17 | 14 | 1 | −2488 | 9 | 31 | 46 | 22 | −3271 | 4 | 9 | 9 | 2 |
| −1823 | 9 | 30 | 41 | 16 | −2563 | 6 | 16 | 19 | 7 | | | | | |

I include also a table of the fields with discriminants not less than $-3280$ possessing a subfield; these are of the form $K(\sqrt{\mu})$ and are obtained in the way outlined in (**2**) and (**3**). It will be seen that the discriminants $-1472$ and $-1984$ appear in both tables.

TABLE 2

| Δ | μ | Δ | μ | Δ | μ |
|---|---|---|---|---|---|
| −275 | $-\frac{1}{2}(1+3\sqrt{5})$ | −1375 | $5+4\sqrt{5}$ | −2000 | $\sqrt{5}$ |
| −400 | $\frac{1}{2}(1+\sqrt{5})$ | −1472 | $-3-4\sqrt{2}$ | −2048 | $\sqrt{2}$ |
| −448 | $-1+2\sqrt{2}$ | −1475 | $\frac{1}{2}(3+7\sqrt{5})$ | −2312 | $\frac{1}{2}(3+\sqrt{17})$ |
| −475 | $-1-2\sqrt{5}$ | −1600 | $1+\sqrt{5}$ | −2375 | $-\frac{1}{2}(5+9\sqrt{5})$ |
| −507 | $-\frac{1}{2}(1+\sqrt{13})$ | −1728 | $3+2\sqrt{3}$ | −2475 | $\frac{1}{2}(3+9\sqrt{5})$ |
| −775 | $-\frac{1}{2}(1+5\sqrt{5})$ | −1775 | $-3-4\sqrt{5}$ | −2704 | $\frac{1}{2}(3+\sqrt{13})$ |
| −1024 | $1+\sqrt{2}$ | −1792 | $1+2\sqrt{2}$ | −3008 | $-5+6\sqrt{2}$ |
| −1156 | $4+\sqrt{17}$ | −1975 | $-\frac{1}{2}(17+11\sqrt{5})$ | −3275 | $-\frac{1}{2}(9+11\sqrt{5})$ |
| −1323 | $\frac{1}{2}(3+\sqrt{21})$ | −1984 | $1+4\sqrt{2}$ | | |

In Delone and Faddeev's table (**1**), 400 is misprinted as 430, and there are three minus signs omitted from the table of coefficients. The coefficients of the polynomials for $\Delta = -491, -775$, and $-848$ should be $(1, 2, 2, -3, -1)$, $(1, 1, 0, 3, -1)$, and $(1, 0, -1, 2, 1)$ respectively.

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# INTEGRAL EQUATIONS FOR ELLIPSOIDAL WAVE FUNCTIONS

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## 1. Introduction

THE special functions of mathematical physics nearly all satisfy certain ordinary linear differential equations of the second order and can conveniently be classified according to the number and nature of the singularities of these equations. This classification can be done in several ways, for example those given by Ince (**1**) and Moon and Spencer (**4**), but for a qualitative assessment of their complexity these functions seem best classified by considering the number of regular singularities in the general equation from which their individual equations can be derived by particularization or by confluence. Thus the simplest are those functions which are of hypergeometric type: that is, those whose equations possess three regular singularities or are derivable by confluence from such an equation; these include Legendre and Bessel functions. The next level of complexity is represented by those functions whose equations are similarly derivable from the Heun equation—that is, the equation with four regular singularities. The next level still, which has been very little explored, is that of functions whose equations are to be derived only from a general equation with five regular singularities; it is to this type that ellipsoidal wave functions belong, for their differential equation has three regular singularities and one irregular singularity.

It is, of course, well known that all functions of hypergeometric type can be expressed as simple definite integrals involving only elementary functions. No such representations have yet been found for functions of Heun type and the conjecture that none such exist (apparently first made by Whittaker as long ago as 1914) now seems well tested. Instead, these functions all satisfy linear homogeneous integral equations: that is, relations of the form

$$f(z) = \lambda \int K(z,z')f(z')\,dz', \tag{1}$$

where the nucleus $K(z,z')$ involves only elementary functions and is (normally) symmetrical in $z$, $z'$. Such equations for the general Heun function were given by Lambe and Ward (**3**), and those for particular

functions of this type (Mathieu functions, Lamé functions, spheroidal wave functions) are well known and can be found in the appropriate parts of (**6**).

For ellipsoidal wave functions, however, no such simple integral equations have yet been given, and it seems probable that none such exist, at least with a symmetric nucleus. The known integral equations satisfied by these functions are of two different types, both involving double integrals: Möglich (**5**) considered equations in two variables of the form

$$f(\alpha)f(\beta) = \lambda \iint K(\alpha,\beta,\alpha',\beta')f(\alpha')f(\beta')\,d\alpha' d\beta', \tag{2}$$

where $K$ is an elementary function, symmetrical in $\alpha$, $\beta$, $\alpha'$, $\beta'$, while Malurkar (**2**) dealt with non-homogeneous equations of the form

$$f(\alpha) = \lambda \iint K(\alpha,\beta,\gamma)f(\beta)f(\gamma)\,d\beta d\gamma, \tag{3}$$

where $K$ is symmetrical in $\alpha$, $\beta$, $\gamma$. In this paper I extend the results of each of these authors and show the relations between them.

*Notation.* By an *ellipsoidal wave function* is meant a solution of the differential equation

$$\frac{d^2w}{dz^2} - (a+bk^2\,\mathrm{sn}^2 z+qk^4\,\mathrm{sn}^4 z)w = 0,$$

possessing the property of being uniform and doubly-periodic in $z$, with periods $2K$ or $4K$, $2iK'$ or $4iK'$. It has been proved by the author in some unpublished work that such functions exist for all values of $q$ and suitably chosen values of $a$ and $b$ (depending on $q$), and that they are in one-to-one correspondence with the Lamé polynomials, to which in fact they reduce when $q = 0$. They fall into eight types, according to their properties of periodicity and parity; I use the general symbol $\mathrm{el}(z)$ to denote any ellipsoidal wave function, and distinguish the eight types by prefixing one or more of the letters $u$, $s$, $c$, $d$ to this, as shown in Table I.

TABLE I

*Classification of ellipsoidal wave functions*

| *Type* | *Parity* | *Periods* | | *Parity* | *Periods* |
|---|---|---|---|---|---|
| $u\,\mathrm{el}(z)$ | even | $2K, 2iK'$ | $sc\,\mathrm{el}(z)$ | odd | $2K, 4iK'$ |
| $s\,\mathrm{el}(z)$ | odd | $4K, 2iK'$ | $sd\,\mathrm{el}(z)$ | odd | $4K, 4iK'$ |
| $c\,\mathrm{el}(z)$ | even | $4K, 4iK'$ | $cd\,\mathrm{el}(z)$ | even | $4K, 2iK'$ |
| $d\,\mathrm{el}(z)$ | even | $2K, 4iK'$ | $scd\,\mathrm{el}(z)$ | odd | $2K, 2iK'$ |

The significance of the letters $u$, $s$, $c$, $d$ is that each ellipsoidal wave

function can be expressed in the form $\operatorname{sn}^r z \operatorname{cn}^s z \operatorname{dn}^t z\, F(\operatorname{sn}^2 z)$, where $r, s, t = 0$ or 1 and $F(\operatorname{sn}^2 z)$ is an integral function of $\operatorname{sn}^2 z$. The prefix letters thus denote those functions—unity, $\operatorname{sn} z$, $\operatorname{cn} z$, $\operatorname{dn} z$—which occur outside $F(\operatorname{sn}^2 z)$, and which characterize the properties: thus a function of type $sd\operatorname{el}(z)$ has the same parity and periods as the function $\operatorname{sn} z \operatorname{dn} z$.

Individual functions of the various types are indicated by adding upper and lower suffix numbers: for instance $u\operatorname{el}_6^2(z)$. The rules for these are immaterial here and are given by the author in (7).

In this paper, the functions will be normalized by the stipulation that the constant factor implicit in the definition shall be so chosen that

$$\iint\limits_S (\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta)\{\operatorname{el}(\alpha)\}^2\{\operatorname{el}(\beta)\}^2\, d\alpha d\beta = i\epsilon, \tag{4}$$

where here and throughout this paper $\iint\limits_S d\alpha d\beta$ denotes integration over the ranges $\alpha = -2K$ to $\alpha = 2K$, $\beta = K-2iK'$ to $\beta = K+2iK'$, the paths of integration being straight lines; $\epsilon$ denotes $\pm 1$, according as $\operatorname{cn} z$ is or is not a factor of the function concerned: that is, $+1$ for functions of types $c\operatorname{el}$, $sc\operatorname{el}$, $cd\operatorname{el}$, $scd\operatorname{el}$, and $-1$ for the others.

It is convenient to denote the product $\operatorname{el}(\alpha)\operatorname{el}(\beta)$, where the 'el' denote precisely the same function, by the single symbol $\operatorname{elp}(\alpha,\beta)$. Where desirable, we prefix the same letters to this as to the el: thus

$$u\operatorname{el}(\alpha)u\operatorname{el}(\beta) = u\operatorname{elp}(\alpha,\beta), \text{ etc.}$$

Before going further, some transformations which will be needed later may conveniently be given here.

'Ellipsoidal coordinates' $\alpha, \beta, \gamma$ are related to Cartesian by the equations

$$\left.\begin{gathered} x = k^2 l \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma, \qquad y = -k^2 k'^{-1} l \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma, \\ z = ik'^{-1} l \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma, \end{gathered}\right\} \tag{5}$$

where the elliptic functions are constructed to modulus $k$, and $l$ is a constant. If the variations in $\alpha$, $\beta$, $\gamma$ are

$$\left.\begin{gathered} \alpha \text{ from } -K+iK' \text{ to } K+iK', \qquad \beta \text{ from } K-iK' \text{ to } K+iK', \\ \gamma \text{ from } -K \text{ to } K, \end{gathered}\right\} \tag{6}$$

then it can be verified that one set of values of $\alpha$, $\beta$, $\gamma$ corresponds to one point $(x, y, z)$ and conversely.

In this coordinate system, the 'wave equation'

$$\nabla^2 V + p^2 V = 0$$

becomes

$$\sum_{\alpha,\beta,\gamma} (\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)\,\partial^2 V/\partial\alpha^2 = k^4 l^2 p^2 (\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)(\mathrm{sn}^2\gamma - \mathrm{sn}^2\alpha)(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta) V. \tag{7}$$

Frequently used, also, is the coordinate system (which has no commonly accepted name but which might suitably be called 'ellipsoidal-polar'):

$$x = kl\,\mathrm{sn}\,\alpha\cos\theta, \quad y = ikl\,\mathrm{cn}\,\alpha\sin\theta\sin\phi, \quad z = il\,\mathrm{dn}\,\alpha\sin\theta\cos\phi, \tag{8}$$

obtainable from (5) by setting

$$\cos\theta = k\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma, \quad \sin\theta\sin\phi = ikk'^{-1}\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma,$$
$$\sin\theta\cos\phi = k'^{-1}\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma, \tag{8 a}$$

in which the variation of $\alpha$ is now from $iK'$ to $K+iK'$, $\theta$ from 0 to $\pi$ and $\phi$ from 0 to $2\pi$, as in the ordinary spherical polar coordinates.

## 2. Integral equations of Möglich's type†

The theorem which is the basis of Möglich's investigations is the following.

(i) *Let* $\iint_S d\alpha' d\beta'$ *denote integration over the ranges* $\alpha'$ *from* $-2K$ *to* $2K$, $\beta'$ *from* $K-2iK'$ *to* $K+2iK'$;

(ii) *let* $F(\alpha,\beta,\alpha',\beta')$ *be a function satisfying the equation*

$$D_{\alpha,\beta}(F) = D_{\alpha',\beta'}(F), \tag{9}$$

*where* $D_{\alpha,\beta}$ *represents the operator*

$$(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)^{-1}(\partial^2/\partial\alpha^2 - \partial^2/\partial\beta^2) - qk^4(\mathrm{sn}^2\alpha + \mathrm{sn}^2\beta),$$

*and also satisfying the conditions*

(*a*) *$F$ is symmetrical in* $\alpha$, $\beta$, $\alpha'$, $\beta'$;

(*b*) *$F$, together with its first two partial derivatives, is bounded and continuous in the range* $S$;

(*c*) *$F$ is doubly-periodic in each variable, its properties of parity and periodicity being the same as the ellipsoidal wave function* $\mathrm{el}(z)$.

*Then* $\mathrm{elp}(\alpha,\beta)$ *satisfies the integral equation*

$$\mathrm{elp}(\alpha,\beta) = \lambda i \iint_S (\mathrm{sn}^2\alpha' - \mathrm{sn}^2\beta') F(\alpha,\beta,\alpha',\beta')\,\mathrm{elp}(\alpha',\beta')\,d\alpha' d\beta'. \tag{10}$$

(The advantage of writing $\lambda i$ instead of the usual $\lambda$ is that under the normalization (4), with $q$ positive, $\lambda$ is real.)

† Möglich's work—the relevant portions of (5) are §§ 4 and 8—is expressed in ellipsoidal-polar coordinates similar to (8 a). Here his results have been transcribed into ellipsoidal coordinates.

The full proof of this (in a quite different notation) is to be found in (5); in outline it is as follows.

From the definition of el($z$) and elp($\alpha,\beta$) it follows that, if $W = \mathrm{elp}(\alpha,\beta)$, then

$$\partial^2 W/\partial\alpha^2 - \partial^2 W/\partial\beta^2 - k^2(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\{b + qk^2(\mathrm{sn}^2\alpha + \mathrm{sn}^2\beta)\}W = 0:$$

that is,
$$\bar{D}_{\alpha,\beta}\, W \equiv (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)(D_{\alpha,\beta} - bk^2)W = 0.$$

Now let

$$G(\alpha,\beta) = \iint\limits_S (\mathrm{sn}^2\alpha' - \mathrm{sn}^2\beta')F(\alpha,\beta,\alpha',\beta')\phi(\alpha',\beta')\, d\alpha' d\beta',$$

where $\phi$ is (so far) arbitrary. Then using condition (ii) above we find that

$$(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)^{-1}\bar{D}_{\alpha,\beta}(G) = I_1 - I_2 + \iint\limits_S F(\alpha,\beta,\alpha',\beta')\bar{D}_{\alpha',\beta'}(\phi)\, d\alpha' d\beta',$$

where
$$I_1 = \int\limits_{K-2iK'}^{K+2iK'} [\phi\,\partial F/\partial\alpha' - F\,\partial\phi/\partial\alpha']_{\alpha'=-2K}^{2K}\, d\beta'$$

and
$$I_2 = \int\limits_{-2K}^{2K} [\phi\,\partial F/\partial\beta' - F\,\partial\phi/\partial\beta']_{\beta'=K-2iK'}^{K+2iK'}\, d\alpha'.$$

Then, if $\phi(\alpha',\beta')$ is specified as $\mathrm{elp}(\alpha',\beta')$, we have $\bar{D}_{\alpha',\beta'}(\phi) = 0$, so that $\bar{D}_{\alpha,\beta}(G) = 0$ also since, by the periodic properties of $\phi$ and $F$, $I_1 = I_2 = 0$ as well. Thus $G(\alpha,\beta)$ satisfies the same partial differential equation as $\mathrm{elp}(\alpha,\beta)$; the conditions imposed on $F$ ensure that it has the same periodic properties as $\mathrm{elp}(\alpha,\beta)$ and from these facts it can be shown that it is none other than a constant multiple of $\mathrm{elp}(\alpha,\beta)$. Thus the theorem is established.

In particular cases the ranges of integration can be reduced; all that is necessary is the vanishing of $I_1$ and $I_2$, and this can be secured by taking the integration over the smallest periods of $\mathrm{elp}(\alpha',\beta')$.

The search for possible nuclei $F(\alpha,\beta,\alpha',\beta')$ thus resolves itself into finding suitable solutions of the equation $D_{\alpha,\beta}(F) = D_{\alpha',\beta'}(F)$, where $D_{\alpha,\beta}$ is given by (9).

As possible nuclei, Möglich obtained the following four solutions of (9):

$$F_1 = \exp\{ihk'^{-2}(\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\alpha'\,\mathrm{dn}\,\beta' - k^3\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\alpha'\,\mathrm{cn}\,\beta')\}, \tag{11 a}$$

$$F_2 = \exp\{hk^2k'^{-1}(\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\alpha'\,\mathrm{cn}\,\beta' - k'\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta')\}, \tag{11 b}$$

$$F_3 = \exp\{ihk'^{-1}(\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\alpha'\,\mathrm{dn}\,\beta' + ik^3k'\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta')\}, \tag{11 c}$$

$$F_4 = \mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta'\, F_1, \tag{11 d}$$

where $h$ denotes $\sqrt{q}$.

The first three he obtains as particular cases of the general result that $\exp(ihR)$ satisfies (9), where

$$R = a\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta' + b\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\alpha'\,\mathrm{cn}\,\beta' + c\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\alpha'\,\mathrm{dn}\,\beta', \tag{12}$$

provided that $a$, $b$, $c$ are such that

$$a^2 - k'^4 b^2 + k^6 = 0, \qquad a^2 - k^4 k'^4 c^2 + k^4 = 0. \tag{13}$$

The proof of this is lengthy but not difficult and can be condensed. It is easily found that, if $F = e^{ihR}$, then

$$F^{-1}D_{\alpha,\beta}(F) = (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)^{-1}[ih(\partial^2 R/\partial\alpha^2 - \partial^2 R/\partial\beta^2) - \\ - q\{(\partial R/\partial\alpha)^2 - (\partial R/\partial\beta)^2\}] - qk^4(\mathrm{sn}^2\alpha + \mathrm{sn}^2\beta).$$

This reduces to

$$F^{-1}D_{\alpha,\beta}(F) = 2k^2 ihR - q\{k^2R^2 - a^2\,\mathrm{sn}^2\alpha'\,\mathrm{sn}^2\beta' + \\ + k'^2 b^2\,\mathrm{cn}^2\alpha'\,\mathrm{cn}^2\beta' - k^2k'^2c^2\,\mathrm{dn}^2\alpha'\,\mathrm{dn}^2\beta' + k^4(\mathrm{sn}^2\alpha + \mathrm{sn}^2\beta)\}.$$

$F^{-1}D_{\alpha',\beta'}(F)$ is a similar expression with $\alpha$, $\beta$ and $\alpha'$, $\beta'$ interchanged; if $F$ satisfies (9), these two expressions must be identical, from which the equations (13) follow without difficulty.

The particular nuclei $F_1$, $F_2$, $F_3$ of (11) are given in turn by putting

$$a = 0, \qquad b = -k^3k'^{-2}, \qquad c = k'^{-2}; \tag{14 a}$$

$$a = ik^2, \qquad b = -ik^2k'^{-1}, \qquad c = 0; \tag{14 b}$$

$$a = ik^3, \qquad b = 0, \qquad c = k'^{-1}. \tag{14 c}$$

Möglich was, apparently, concerned only with developing at least one integral equation which would be satisfied by ellipsoidal wave-function products of each of the eight types. (The above three nuclei provide equations for all types except *scd* elp; it was to fill this gap that he obtained $F_4$). Nor was he concerned that each of the above nuclei is satisfied by functions of more than one type because of the different periodicity of the two parts of each nucleus. Here, however, we are concerned with a more systematic development of the possible nuclei, and it is convenient to form, from $F_1$, $F_2$, $F_3$, other nuclei which provide equations satisfied only by functions of one type.

We observe that in (14) any of the signs could be changed in $a$, $b$, $c$ and still satisfy (13); by taking sums and differences of the results, we can obtain other solutions of (9). Thus, for instance, from the two nuclei

$$\exp\{ihk'^{-2}(\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\alpha'\,\mathrm{dn}\,\beta' \pm k^3\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\alpha'\,\mathrm{cn}\,\beta'\}$$

we construct the four further nuclei

$$\frac{\cos}{\sin}(hk'^{-2}\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\alpha'\,\mathrm{dn}\,\beta')\frac{\cos}{\sin}(hk^3k'^{-2}\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\alpha'\,\mathrm{cn}\,\beta').$$

Four further nuclei can similarly be constructed from each of $F_2$ and $F_3$.

The type of ellipsoidal wave function to be associated with each of the nuclei can be found by considering parity and periodicity since these must be the same for the nucleus as for the function. Carrying out this process, we find the following nuclei $F(\alpha, \beta, \alpha', \beta')$ for the equation (10); for brevity I write $S$ for $\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta'$, etc.

*Functions of type u* elp.

$$\cos(hk'^{-2}D)\cos(hk^3k'^{-2}C), \tag{15a}$$

$$\cosh(hk^2k'^{-1}C)\cosh(hk^2S), \tag{15b}$$

$$\cos(hk'^{-1}D)\cosh(hk^3S). \tag{15c}$$

*Type s* elp.

$$\cosh(hk^2k'^{-1}C)\sinh(hk^2S), \tag{15d}$$

$$\cos(hk'^{-1}D)\sinh(hk^3S). \tag{15e}$$

*Type c* elp.

$$\cos(hk'^{-2}D)\sin(hk^3k'^{-2}C), \tag{15f}$$

$$\sinh(hk^2k'^{-1}C)\cosh(hk^2S). \tag{15g}$$

*Type d* elp.

$$\sin(hk'^{-2}D)\cos(hk^3k'^{-2}C), \tag{15h}$$

$$\sin(hk'^{-1}D)\cosh(hk^3S). \tag{15i}$$

*Type sc* elp.

$$\sinh(hk^2k'^{-1}C)\sinh(hk^2S). \tag{15j}$$

*Type sd* elp.

$$\sin(hk'^{-1}D)\sinh(hk^3S). \tag{15k}$$

*Type cd* elp.

$$\sin(hk'^{-2}D)\sin(hk^3k'^{-2}C). \tag{15l}$$

We have thus nuclei for functions of each type except *scd* elp. For these we are led (as was Möglich) to seek a nucleus of the form $Ue^{ihR}$, but before doing so it is interesting to inquire whether it is possible to find further nuclei of the form $e^{ihR}$.

This can conveniently be discussed by going over to polar coordinates as in (8a), with $\alpha$ instead of $\gamma$: that is,

$$\cos\theta = k\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta, \qquad \cos\theta' = k\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta', \tag{16}$$

and so forth, whereby (9) becomes

$$\partial^2F/\partial\theta^2+\cot\theta\,\partial F/\partial\theta+\mathrm{cosec}^2\theta\,\partial^2F/\partial\phi^2+q(\cos^2\theta+k'^2\sin^2\theta\cos^2\phi)F$$

$$= \text{(the same expression with } \theta', \phi' \text{ for } \theta, \phi). \tag{17}$$

If we set $F = e^{ihR}$ in this, it becomes

$$ih[\partial^2 R/\partial\theta^2 + \cot\theta\,\partial R/\partial\theta + \operatorname{cosec}^2\theta\,\partial^2 R/\partial\phi^2] -$$
$$-q[(\partial R/\partial\theta)^2 + \operatorname{cosec}^2\theta(\partial R/\partial\phi)^2 - \cos^2\theta - k'^2\sin^2\theta\cos^2\phi]$$
$$= (\text{the same expression with } \theta', \phi' \text{ for } \theta, \phi). \quad (18\,\mathrm{a})$$

Let us write this more briefly as

$$ihD_1(R) - qD_2(R) = ihD_1'(R) - qD_2'(R). \quad (18\,\mathrm{b})$$

Now, if $Y(\theta,\phi)$ is any spherical surface harmonic of degree $n$, we have

$$\partial^2 Y/\partial\theta^2 + \cot\theta\,\partial Y/\partial\theta + \operatorname{cosec}^2\theta\,\partial^2 Y/\partial\phi^2 + n(n+1)Y = 0,$$

and consequently, if we take $R = Y(\theta,\phi)Y(\theta',\phi')$, or any linear combination of such expressions, then $D_1(R) = D_1'(R)$. Further nuclei of the form $e^{ihR}$ will therefore be provided by

$$R = \sum_n \sum_m A_n^m Y_n^m(\theta,\phi) Y_n^m(\theta',\phi')$$

if the $A_n^m$ are so chosen that $D_2(R) = D_2'(R)$: that is,

$$(\partial R/\partial\theta)^2 + \operatorname{cosec}^2\theta(\partial R/\partial\phi)^2 - \cos^2\theta - k'^2\sin^2\theta\cos^2\phi$$
$$= (\partial R/\partial\theta')^2 + \operatorname{cosec}^2\theta'(\partial R/\partial\phi')^2 - \cos^2\theta' - k'^2\sin^2\theta'\cos^2\phi'. \quad (19)$$

It is clear that the single surface harmonic of degree zero, namely 1, does not satisfy (19). Combinations of the three surface harmonics of degree 1, namely $\cos\theta$, $\sin\theta\sin\phi$, $\sin\theta\cos\phi$, provide in effect the nuclei $F_1$, $F_2$, $F_3$ given in (11), the values of $a$, $b$, $c$ necessary to satisfy (19) being precisely those determined by (13). It might be expected that one could form a linear combination of the five surface harmonics of degree 2 in such a way as to satisfy (19), but on investigation this does not appear to be the case, nor is it possible to form a suitable linear combination of the nine harmonics of degrees 0, 1, and 2. Consideration of the higher harmonics becomes very complicated, and I have not gone into them.

An interesting generalization of the nuclei $F_1$, $F_2$, $F_3$ has been given by Möglich in (5) § 13; it is obtained by taking (12) with

$$a = \pm ik^3 \operatorname{sn}\gamma, \qquad b = \pm k^3 k'^{-2} \operatorname{cn}\gamma, \qquad c = \pm k'^{-2} \operatorname{dn}\gamma,$$

where $\gamma$ is arbitrary. By suitable combination of the $\pm$ signs, we obtain eight further nuclei each of which is associated with one type of ellipsoidal wave function:

$$\begin{matrix}\cosh\\ \sinh\end{matrix}(hk^3 \operatorname{sn}\gamma\, S) \begin{matrix}\cos\\ \sin\end{matrix}(hk^3k'^{-2} \operatorname{cn}\gamma\, C) \begin{matrix}\cos\\ \sin\end{matrix}(hk'^{-2} \operatorname{dn}\gamma\, D), \quad (20)$$

where $S = \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\alpha' \operatorname{sn}\beta'$, etc., as before.

We turn now to the question of finding nuclei of the form $Ue^{ihR}$, where $R$ is itself a nucleus. With little difficulty it is found that a necessary and sufficient condition is that

$$(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)^{-1}\left[\frac{\partial^2 U}{\partial\alpha^2}-\frac{\partial^2 U}{\partial\beta^2}+2ihR\left\{\frac{\partial U}{\partial\alpha}\frac{\partial R}{\partial\alpha}-\frac{\partial U}{\partial\beta}\frac{\partial R}{\partial\beta}\right\}\right]$$
$$= \text{(the same expression with } \alpha', \beta' \text{ for } \alpha, \beta), \quad \text{(21 a)}$$

which, in the $\theta$, $\phi$ notation, becomes

$$\frac{\partial^2 U}{\partial\theta^2}+\cot\theta\frac{\partial U}{\partial\theta}+\operatorname{cosec}^2\theta\frac{\partial^2 U}{\partial\phi^2}+2ihR\left\{\frac{\partial U}{\partial\theta}\frac{\partial R}{\partial\theta}+\operatorname{cosec}^2\theta\frac{\partial U}{\partial\phi}\frac{\partial R}{\partial\phi}\right\}$$
$$= \text{(the same expression with } \theta', \phi' \text{ for } \theta, \phi). \quad \text{(21 b)}$$

Clearly, then, we may take $U$ as any expression $Y(\theta,\phi)Y(\theta',\phi')$, or a linear combination of such expressions, if it can be made to satisfy the condition

$$\frac{\partial U}{\partial\theta}\frac{\partial R}{\partial\theta}+\operatorname{cosec}^2\theta\frac{\partial U}{\partial\phi}\frac{\partial R}{\partial\phi}=\frac{\partial U}{\partial\theta'}\frac{\partial R}{\partial\theta'}+\operatorname{cosec}^2\theta'\frac{\partial U}{\partial\phi'}\frac{\partial R}{\partial\phi'}. \quad (22)$$

Taking

$$R = A\cos\theta\cos\theta'+B\sin\theta\sin\theta'\sin\phi\sin\phi'+C\sin\theta\sin\theta'\cos\phi\cos\phi',$$

we easily verify that we can have

$$U = \cos\theta\cos\theta' \quad \text{(provided that } A=0),$$
$$U = \sin\theta\sin\theta'\sin\phi\sin\phi' \quad \text{(provided that } B=0),$$
$$U = \sin\theta\sin\theta'\cos\phi\cos\phi' \quad \text{(provided that } C=0).$$

These, when we revert to the $\alpha$, $\beta$, $\alpha'$, $\beta'$, lead to the further nuclei

$$\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\alpha'\,\mathrm{sn}\,\beta'\,F_1, \quad \text{(23 a)}$$
$$\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\alpha'\,\mathrm{dn}\,\beta'\,F_2, \quad \text{(23 b)}$$
$$\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\alpha'\,\mathrm{cn}\,\beta'\,F_3, \quad \text{(23 c)}$$

where $F_1$, $F_2$, $F_3$ have the meanings given in (11). As has been mentioned, the first of these was given by Möglich, who obtained it by a different process [(**5**), § 8.]

Consequently, from the twelve nuclei in (15) we obtain another twelve by multiplying each by that factor $C$, $D$, or $S$ which does not occur already. These are allocated to the various types of function in the same way, and we thus have the further nuclei of equation (10).

*Type s* elp. $\quad S\cos(hk'^{-2}D)\cos(hk^3k'^{-2}C)$. (24 a)

*Type c* elp. $\quad C\cos(hk'^{-1}D)\cosh(hk^3S)$. (24 b)

*Type d* elp. $\quad D\cosh(hk^2k'^{-1}C)\cosh(hk^2S)$. (24 c)

*Type sc* elp. $S\cos(hk'^{-2}D)\sin(hk^3k'^{-2}C)$, (24 d)

$C\cos(hk'^{-1}D)\sinh(hk^3S)$. (24 e)

*Type sd* elp. $S\sin(hk'^{-2}D)\cos(hk^3k'^{-2}C)$, (24 f)

$D\cosh(hk^2k'^{-1}C)\sinh(hk^2S)$. (24 g)

*Type cd* elp. $D\sinh(hk^2k'^{-1}C)\cosh(hk^2S)$, (24 h)

$C\sin(hk'^{-1}D)\cosh(hk^3S)$. (24 i)

*Type scd* elp. $S\sin(hk'^{-2}D)\sin(hk^3k'^{-2}C)$, (24 j)

$D\sinh(hk^2k'^{-1}C)\sinh(hk^2S)$, (24 k)

$C\sin(hk'^{-1}D)\sinh(hk^3S)$. (24 l)

The formulae (15) and (24) thus provide in all three nuclei for each type of ellipsoidal wave function, for the general integral equation (10).

There does not seem, however, to be any simple way of writing down further nuclei for this equation: that is, solutions of (9). This is in contrast to the integral equations of Malurkar's type for which a large number of nuclei can easily be obtained.

## 3. Integral equations of Malurkar's type

The integral equations for ellipsoidal wave functions given by Malurkar are of a different type from those of Möglich, and, while particular cases of the former can be deduced from particular cases of the latter, it does not seem true to say that Malurkar's equations are a transformation of Möglich's. The theorem leading to Malurkar's equations is

(i) *Let* $\iint_S d\beta d\gamma$ *denote integration over the ranges* $\beta = -2K$ *to* $2K$, $\gamma = K-2iK'$ *to* $K+2iK'$;

(ii) *let* $f(\alpha,\beta,\gamma)$ *be a function satisfying the equation*

$$\sum_{\alpha,\beta,\gamma}(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)\,\partial^2 f/\partial\alpha^2 = -qk^4(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)f \tag{25}$$

*and also the conditions*

(*a*) *f is symmetrical in* $\alpha$, $\beta$, $\gamma$;

(*b*) *f, together with its first two partial derivatives, is bounded and continuous in the range* $S$;

(*c*) *f is doubly-periodic in each variable, its properties of periodicity and parity being the same as the ellipsoidal wave function* $\mathrm{el}(z)$.

*Then* $\mathrm{el}(z)$ *satisfies the integral equation*

$$\mathrm{el}(\alpha) = \lambda i \iint_S (\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)f(\alpha,\beta,\gamma)\,\mathrm{elp}(\beta,\gamma)\,d\beta d\gamma. \tag{26}$$

It may be noted that, since the equation is not homogeneous, the value of $\lambda$ is not independent of the normalization. Here we shall normalize by equation (4).

The proof of this is given in full by Malurkar [(**2**), part III ($a$)], and follows the usual lines of differentiation under the integral sign with respect to $\alpha$, followed by partial integration with respect to $\beta$ and $\gamma$.

The importance of Malurkar's equations is that, when (25) is compared with (7), they are found to be identical except for adjustment of the constants; thus (25) is simply the wave equation $\nabla^2 f = -p^2 f$ under the transformation

$$px = ik^2 h \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma, \qquad py = -ik^2 k'^{-1} h \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma,$$
$$pz = -hk'^{-1} \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma, \tag{27}$$

where $h = \sqrt{q}$ as before. Consequently any bounded, continuously differentiable solution of the wave equation, when expressed in terms of $\alpha, \beta, \gamma$, will yield a nucleus $f(\alpha, \beta, \gamma)$ of the equation (26). Now we have the simple solutions of $\nabla^2 f = -p^2 f$,

$$e^{ipx},\ e^{ipy},\ e^{ipz},$$
$$ye^{ipx},\ ze^{ipx},\ xe^{ipy},\ ze^{ipy},\ xe^{ipz},\ ye^{ipz},$$
$$yze^{ipx},\ xze^{ipy},\ xye^{ipz}.$$

The same expressions with $-p$ written for $p$ will also be solutions, and, by forming the sums and differences, e.g. $e^{ipx} \pm e^{-ipx}$, we obtain a total of twenty-four simple nuclei of (26) which we allocate to the different types of ellipsoidal wave function by considering parity and periodicity as before. These nuclei are as follows; for brevity I write $S$, $C$, $D$ for $\operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma$, $\operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma$, $\operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma$, respectively.

*Type u* el.

$\cosh hk^2 S$, (28 a)

$\cosh hk^2 k'^{-1} C$, (28 b)

$\cos hk'^{-1} D$. (28 c)

*Type s* el.

$\sinh hk^2 S$, (28 d)

$S \cosh hk^2 k'^{-1} C$, (28 e)

$S \cos hk'^{-1} D$. (28 f)

*Type c* el.

$\sinh hk^2 k'^{-1} C$, (28 g)

$C \cosh hk^2 S$, (28 h)

$C \cos hk'^{-1} D$. (28 i)

*Type d* el.

$\sin hk'^{-1} D$, (28 j)

$D \cosh hk^2 S$, (28 k)

$D \cosh hk^2 k'^{-1} C$. (28 l)

*Type sc* el.

$C \sinh hk^2 S$, (28 m)

$S \sinh hk^2 k'^{-1} C$, (28 n)

$SC \cos hk'^{-1} D$. (28 o)

*Type sd* el.

$D \sinh hk^2 S$, (28 p)

$S \sin hk'^{-1} D$, (28 q)

$SD \cosh hk^2 k'^{-1} C$. (28 r)

*Type cd* el.

$D\sinh hk^2k'^{-1}C,$ (28 s)

$C\sin hk'^{-1}D,$ (28 t)

$CD\cosh hk^2S.$ (28 u)

*Type scd* el.

$CD\sinh hk^2S,$ (28 v)

$SD\sinh hk^2k'^{-1}C,$ (28 w)

$SC\sin hk'^{-1}D.$ (28 x)

Of these, types (a), (d), (g), (j), (m), (q), (s), (x) only were given by Malurkar, who apparently overlooked the general nature of equation (25) since he states that these eight are the only forms of nuclei. Obviously, by taking other solutions of the wave equation, we can obtain many other nuclei in addition to those above.

## 4. Relations between the two sets of equations

These two sets of equations do not seem to be derivable, in their entirety, the one from the other. However, some of the simpler equations of Malurkar's type are easily derived from the corresponding equations of Möglich's type. For example, if we take equation (9) with nucleus (15 a) and set $\beta = K+iK'$, then

$$\operatorname{dn}\beta = 0, \quad \operatorname{cn}\beta = -ik'k^{-1},$$

and it becomes

$$u\,\mathrm{el}(\alpha)\,u\,\mathrm{el}(K+iK')$$
$$= \lambda i \iint_S (\operatorname{sn}^2\alpha' - \operatorname{sn}^2\beta')\cosh(hk^2k'^{-1}\operatorname{cn}\alpha\operatorname{cn}\alpha'\operatorname{cn}\beta')u\,\mathrm{elp}(\alpha',\beta')\,d\alpha'd\beta',$$

which is the equation (26) with nucleus (28 b) when we write $\beta$, $\gamma$ for $\alpha'$, $\beta'$ and adjust the characteristic constant. Similarly, by giving to $\beta$ the values 0, $K$, or $K+iK'$, we can derive one or more equations of Malurkar's type from one or more of Möglich's type. The correspondence is set out in the following table.

TABLE II

*Correspondence between nuclei of equations* (9) *and* (26)

(The first column shows the nucleus or nuclei of Möglich's type (15) or (24), the second column the nucleus or nuclei of Malurkar's type (28) which can be derived from the former.)

| | | | | | |
|---|---|---|---|---|---|
| (15 a) | (28 b, c) | (24 a) | (28 e, f) | (24 g) | (28 p) |
| (15 b) | (28 a, b) | (24 b) | (28 h, i) | (24 h) | (28 s) |
| (15 c) | (28 a, c) | (24 c) | (28 k, l) | (24 i) | (28 t) |
| (15 d, e) | (28 d) | (24 d) | (28 n) | | |
| (15 f, g) | (28 g) | (24 e) | (28 m) | | |
| (15 h, i) | (28 j) | (24 f) | (28 q) | | |

Thus all the nuclei (28) except (o), (r), (u), (v), (w), (x), correspond to nuclei (15) or (24): that is, with the exception of those of (28) with more

than one of the factors $S$, $C$, $D$ before the trigonometric or hyperbolic term. It is noteworthy that none of the nuclei for functions of the type *scd* el are thus connected; this suggests that the derivation of equations of Malurkar's type from equations of Möglich's type can be effected only in simple cases.

## 5. Determination of the value of $\lambda$

The problem of finding the eigenvalues of the parameter $\lambda$ can be approached in two ways. Malurkar [(2) 67–68] has shown how a series can be obtained for $\lambda$ in ascending powers of $q$, valid when $q$ is sufficiently small, by assuming such a series and also a similar series for the ellipsoidal wave functions themselves: that is, the normal 'perturbation' type of solution. For general values of $q$, it is possible to find expressions for $\lambda$ in terms of particular values of the eigenfunctions concerned, but this depends on the expansion of both the nucleus and the elp-function as series of Lamé polynomial products. This involves considerable preliminary work, of importance for its own sake, and will be treated by the author in a subsequent paper.

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# ON THE CONVERGENCE OF EIGENFUNCTION EXPANSIONS (II)

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**1.** CONSIDER the eigenfunction problem arising from the differential equation

$$\frac{d^2\phi}{dx^2}+\{\lambda-q(x)\}\phi = 0 \quad (0 \leqslant x < \infty) \tag{1.1}$$

with the boundary condition

$$\phi(0)\cos\alpha+\phi'(0)\sin\alpha = 0. \tag{1.2}$$

If $q(x) \to \infty$ as $x \to \infty$, there are discrete eigenvalues $\lambda_n$ with corresponding eigenfunctions $\psi_n(x)$. Let $f(t)$ be a given function of $L^2(0,\infty)$ and let

$$c_n = \int\limits_0^\infty f(t)\psi_n(t)\,dt.$$

The eigenfunction expansion of $f(x)$ is then

$$f(x) = \sum_{n=0}^{\infty} c_n\,\psi_n(x). \tag{1.3}$$

In my previous paper (**2**) I proved the following theorem: *Let $q(x)$ be twice differentiable, $q'(x) > 0$, $q''(x) \geqslant 0$, $q''(x) \leqslant \{q'(x)\}^\gamma$ for sufficiently large values of $x$, where $1 < \gamma < \frac{4}{3}$. Let $f(t)$ be $L^2(0,\infty)$ and, in the neighbourhood of $t = x$, let $f(t)$ satisfy any condition which is sufficient for the convergence of an ordinary Fourier series. Then* (1.3) *holds in the sense of ordinary convergence.*

The conditions on $q(x)$ were assumed so that certain theorems on the distribution of the eigenvalues proved in my book *Eigenfunction Expansions* under these conditions could be applied. Subsequent research has shown such conditions to be unnecessary. In particular it has been proved by Hartman (**1**) that, if $N(\lambda)$ is the number of eigenvalues not exceeding $\lambda$, then the formula

$$N(\lambda) = \frac{1}{\pi}\int\limits_0^p \{\lambda-q(x)\}^{\frac{1}{2}}\,dx + O(1), \tag{1.4}$$

where $q(p) = \lambda$, is true for any function $q(x)$ which is continuous, increasing, and convex downwards.

I shall now show that the convergence theorem is true under the same conditions. It therefore now stands as follows.

*Let $q(x)$ be continuous, increasing, and convex downwards. Let $f(t)$ be $L^2(0,\infty)$ and, in the neighbourhood of $t = x > 0$, let $f(t)$ satisfy any condition which is sufficient for the convergence of an ordinary Fourier series. Then* (1.3) *holds in the sense of ordinary convergence.*

In the following analysis we suppose also that $q(0) = 0$. This does not involve any loss of generality since, if $q(0) \neq 0$, we can replace $q(x)$ by $q(x)-q(0)$ and $\lambda$ by $\lambda-q(0)$ without altering the problem.

**2.** Let $x = p(\lambda)$ be the function inverse to $q(x) = \lambda$ and let $p_n = p(\lambda_n)$.

Since
$$\int_0^p \{\lambda-q(x)\}^{\frac{1}{2}}\,dx \leqslant p\lambda^{\frac{1}{2}},$$
we obtain, on putting $\lambda = \lambda_n$ in (1.4),
$$n < Ap_n\lambda_n^{\frac{1}{2}}. \tag{2.1}$$
Also (1.4) gives
$$\begin{aligned}\pi\{N(\lambda+\surd\lambda)-N(\lambda)\}\\ &= \int_0^{p(\lambda+\surd\lambda)} \{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}\,dx - \int_0^{p(\lambda)} \{\lambda-q(x)\}^{\frac{1}{2}}\,dx + O(1)\\ &= \int_0^{p(\lambda)} [\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}-\{\lambda-q(x)\}^{\frac{1}{2}}]\,dx+\\ &\qquad + \int_{p(\lambda)}^{p(\lambda+\surd\lambda)} \{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}\,dx+O(1)\\ &= I_1+I_2+O(1),\end{aligned}$$
say. Now
$$\begin{aligned}\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}-\{\lambda-q(x)\}^{\frac{1}{2}} &= \frac{\surd\lambda}{\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}+\{\lambda-q(x)\}^{\frac{1}{2}}}\\ &\leqslant \frac{\surd\lambda}{\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}}.\end{aligned}$$
On account of the convexity of $q(x)$, if $x \leqslant p(\lambda)$, then
$$\frac{q(x)}{x} \leqslant \frac{q\{p(\lambda)\}}{p(\lambda)} = \frac{\lambda}{p(\lambda)}.$$
Hence
$$\lambda+\surd\lambda-q(x) \geqslant \lambda+\surd\lambda-\lambda x/p(\lambda).$$
Hence
$$\begin{aligned}I_1 \leqslant \int_0^{p(\lambda)} \frac{\surd\lambda\,dx}{\{\lambda+\surd\lambda-\lambda x/p(\lambda)\}^{\frac{1}{2}}} &= -\frac{2p(\lambda)}{\surd\lambda}\left[\left\{\lambda+\surd\lambda-\frac{\lambda x}{p(\lambda)}\right\}^{\frac{1}{2}}\right]_0^{p(\lambda)}\\ &= O\{p(\lambda)\}.\end{aligned}$$

Since $p(\lambda)$ is concave downwards and $p(0) = 0$, if $\mu > 0$,

$$\frac{p(\lambda)}{\lambda} \geqslant \frac{p(\lambda+\mu)}{\lambda+\mu}.$$

Hence $$p(\lambda+\mu)-p(\lambda) \leqslant \mu\, p(\lambda)/\lambda,$$

and so $$I_2 = O[\{p(\lambda+\surd\lambda)-p(\lambda)\}\lambda^{\frac{1}{4}}] = O\{\lambda^{-\frac{1}{4}}\, p(\lambda)\}.$$

Altogether $$N(\lambda+\surd\lambda)-N(\lambda) = O\{p(\lambda)\}. \tag{2.2}$$

In particular $$N(\lambda_n+\surd\lambda_n)-N(\lambda_n) = O(p_n). \tag{2.3}$$

**3.** We shall next prove that, *for any fixed positive $x$, as $n \to \infty$*

$$\psi_n(x) = O(p_n^{-\frac{1}{4}}). \tag{3.1}$$

This is Lemma 9.8 of my book. The following proof is more general than that in the book since it does not assume that $q(x)$ is differentiable, and is also simpler.

We have $$\psi_n''(x)+\{\lambda_n-q(x)\}\psi_n(x) = 0.$$

The coefficient of $\psi_n(x)$ is positive if $0 < x < p_n$. Hence in this range $\psi_n(x)$ is concave downwards where it is positive and upwards where it is negative, and it has just one maximum or minimum between consecutive zeros.

Let $$F(x) = \psi_n^2(x)+\psi_n'^2(x)/\{\lambda_n-q(x)\}.$$

Then, if $0 \leqslant a < b < p_n$,

$$\int_a^b \psi_n'^2(x)\, d\left\{\frac{1}{\lambda_n-q(x)}\right\} = \left[\frac{\psi_n'^2(x)}{\lambda_n-q(x)}\right]_a^b - \int_a^b \frac{2\psi_n'(x)\psi_n''(x)}{\lambda_n-q(x)}\, dx$$

$$= \left[\frac{\psi_n'^2(x)}{\lambda_n-q(x)}\right]_a^b + 2\int_a^b \psi_n'(x)\psi_n(x)\, dx$$

$$= \left[\frac{\psi_n'^2(x)}{\lambda_n-q(x)}+\psi_n^2(x)\right]_a^b = F(b)-F(a).$$

The left-hand side is non-negative, and so $F(x)$ is non-decreasing.

Consider first the case where $\psi_n(0) = 0$. We have

$$\int_0^X \psi_n'^2(x)\, dx = [\psi_n(x)\psi_n'(x)]_0^X - \int_0^X \psi_n(x)\psi_n''(x)\, dx$$

$$= [\psi_n(x)\psi_n'(x)]_0^X - \int_0^X \psi_n^2(x)\{q(x)-\lambda_n\}\, dx.$$

The integrated term vanishes at $x = 0$ and tends to 0 at infinity (see

Lemma 10.2 of my book). Hence

$$\int_0^\infty \{\psi_n'^2(x)+q(x)\psi_n^2(x)\}\,dx = \lambda_n \int_0^\infty \psi_n^2(x)\,dx = \lambda_n. \tag{3.2}$$

Thus
$$\int_0^\infty \psi_n'^2(x)\,dx < \lambda_n. \tag{3.3}$$

Since $q(x)$ is convex downwards,

$$q(\tfrac{1}{2}x) \leqslant \tfrac{1}{2}\{q(0)+q(x)\} = \tfrac{1}{2}q(x).$$

Hence
$$q(\tfrac{1}{2}p_n) \leqslant \tfrac{1}{2}q(p_n) = \tfrac{1}{2}\lambda_n.$$

Accordingly, if $x \leqslant \frac{1}{2}p_n$,

$$F(x) \leqslant \psi_n^2(x)+2\psi_n'^2(x)/\lambda_n.$$

It follows that, if $0 \leqslant \xi < \frac{1}{2}p_n$,

$$\int_\xi^{\frac{1}{2}p_n} F(x)\,dx \leqslant \int_\xi^{\frac{1}{2}p_n} \{\psi_n^2(x)+2\psi_n'^2(x)/\lambda_n\}\,dx$$
$$\leqslant \int_0^\infty \{\psi_n^2(x)+2\psi_n'^2(x)/\lambda_n\}\,dx < 3.$$

Hence, since $F(x)$ is non-decreasing,

$$(\tfrac{1}{2}p_n-\xi)F(\xi) < 3.$$

Thus $F(x) = O(p_n^{-1})$ uniformly in any fixed interval $x_1 \leqslant x \leqslant x_2$. If $x$ is a zero of $\psi_n'(x)$ in this interval, $F(x) = \psi_n^2(x)$, so that

$$\psi_n(x) = O(p_n^{-1}).$$

Such values of $x$ are maxima or minima of $\psi_n(x)$, so that (3.1) follows.

In the case of the general boundary condition (1.2), (3.2) is replaced by

$$\int_0^\infty \{\psi_n'^2(x)+q(x)\psi_n^2(x)\}\,dx = \lambda_n-\psi_n(0)\psi_n'(0)$$
$$= \lambda_n+\psi_n^2(0)\cot\alpha.$$

If $\cot\alpha \leqslant 0$, the result follows as before. If $\cot\alpha > 0$, the result follows if $\psi_n(0) = O(\lambda_n^{\frac{1}{4}})$. Otherwise let $x_0$ be the smallest zero of $\psi_n(x)$. If $\psi_n(0) > 0$, then $\psi_n'(0) < 0$, and $\psi_n(x)$ is positive and concave downwards over $(0, x_0)$ if $x_0 < p_n$. It follows that

$$\psi_n(x) \geqslant (1-x/x_0)\psi_n(0)$$

in $(0, x_0)$. Hence

$$\int_0^{x_0} \psi_n^2(x)\,dx \geqslant \psi_n^2(0) \int_0^{x_0} \left(1-\frac{x}{x_0}\right)^2 dx = \tfrac{1}{3}\psi_n^2(0)x_0.$$

Hence $$\psi_n^2(0)x_0 \leqslant 3.$$

If $\psi_n(0) > A\lambda_n^{\frac{1}{4}}$, this implies that $x_0 < A/\lambda_n$. Now

$$\int_{x_0}^{\infty} \{\psi_n'^2(x)+q(x)\psi_n^2(x)\}\,dx = \lambda_n \int_{x_0}^{\infty} \psi_n^2(x)\,dx,$$

and the result again follows since $x_0 < x_1$ for $n$ large enough.

The proof of the main theorem is now similar to that given in my previous paper.

## REFERENCES

1. P. Hartman, 'On the zeros of solutions of second-order linear differential equations', *J. London Math. Soc.* 27 (1952) 492–6.
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